Markovian Mpemba and Pontus-Mpemba effects in open nonequilibrium quantum systems

Capri Spring School

WORLD QUANTUM DAY
APRIL 14

Heinrich Heine Universität Düsseldorf

CRC 183 ENTANGLED STATES OF MATTER

ML4Q MATTER AND LIGHT FOR QUANTUM COMPUTING

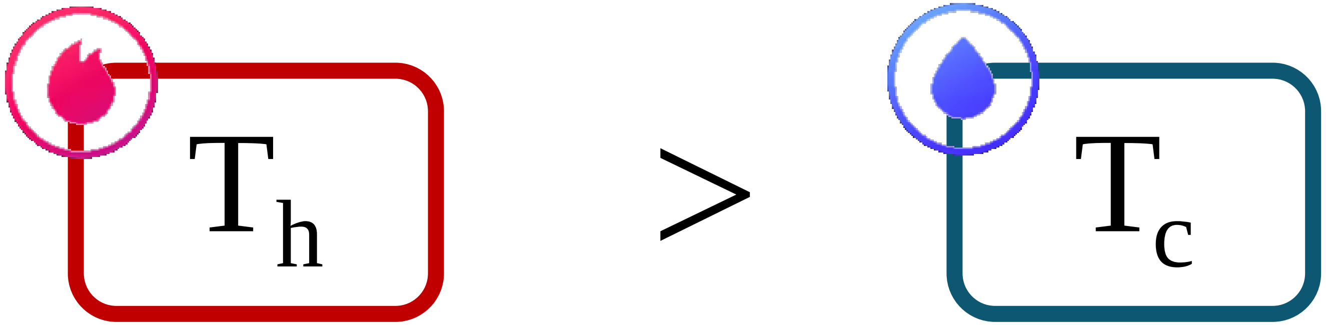
The roadmap

- THE MPEMBA EFFECT:
  - From Aristoteles to Mpemba, i.e., making ice cream in Tanzania
  - Open non-equilibrium quantum systems

- THE PONTUS MPEMBA EFFECT:
  - Fishing with Aristoteles
  - Relaxation protocols

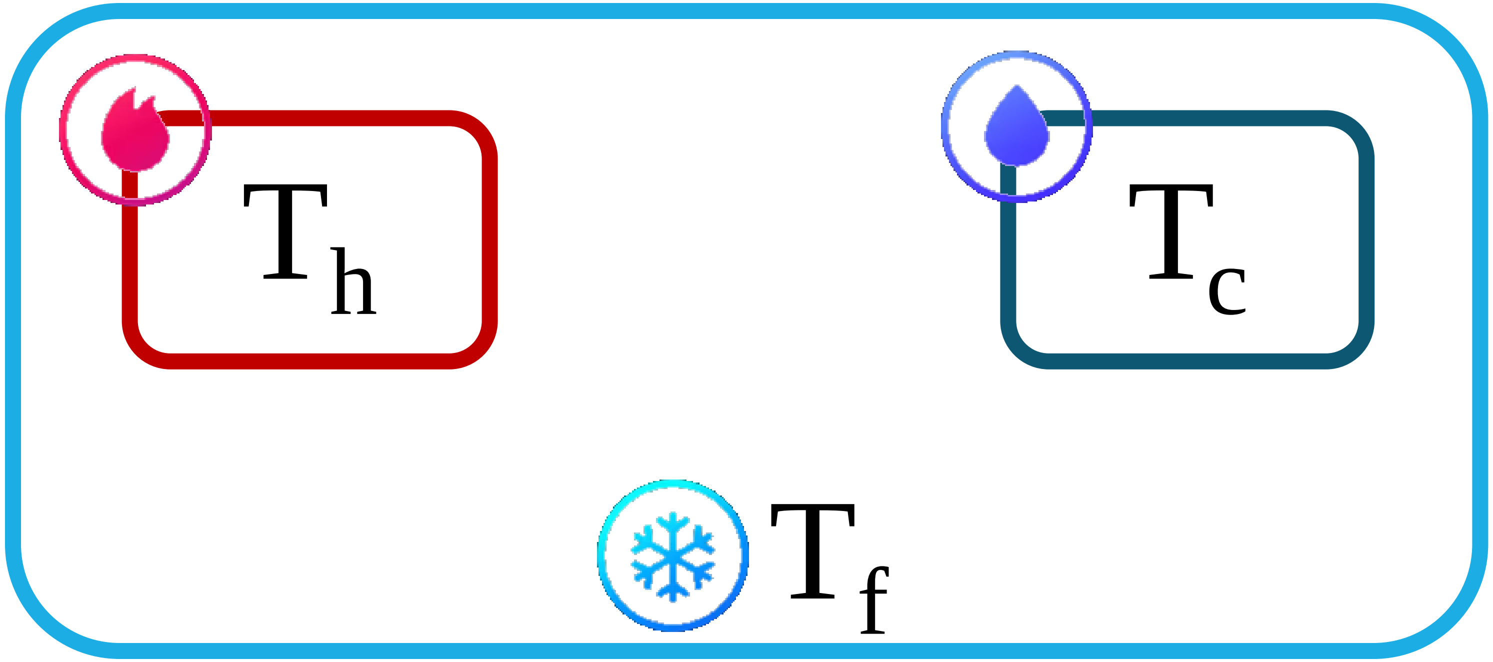
ICE CREAM IN TANZANIA

- Prepare two copies of the same system (a bucket of water) in the thermal state at temperature T_h (hot) and T_c (cold)

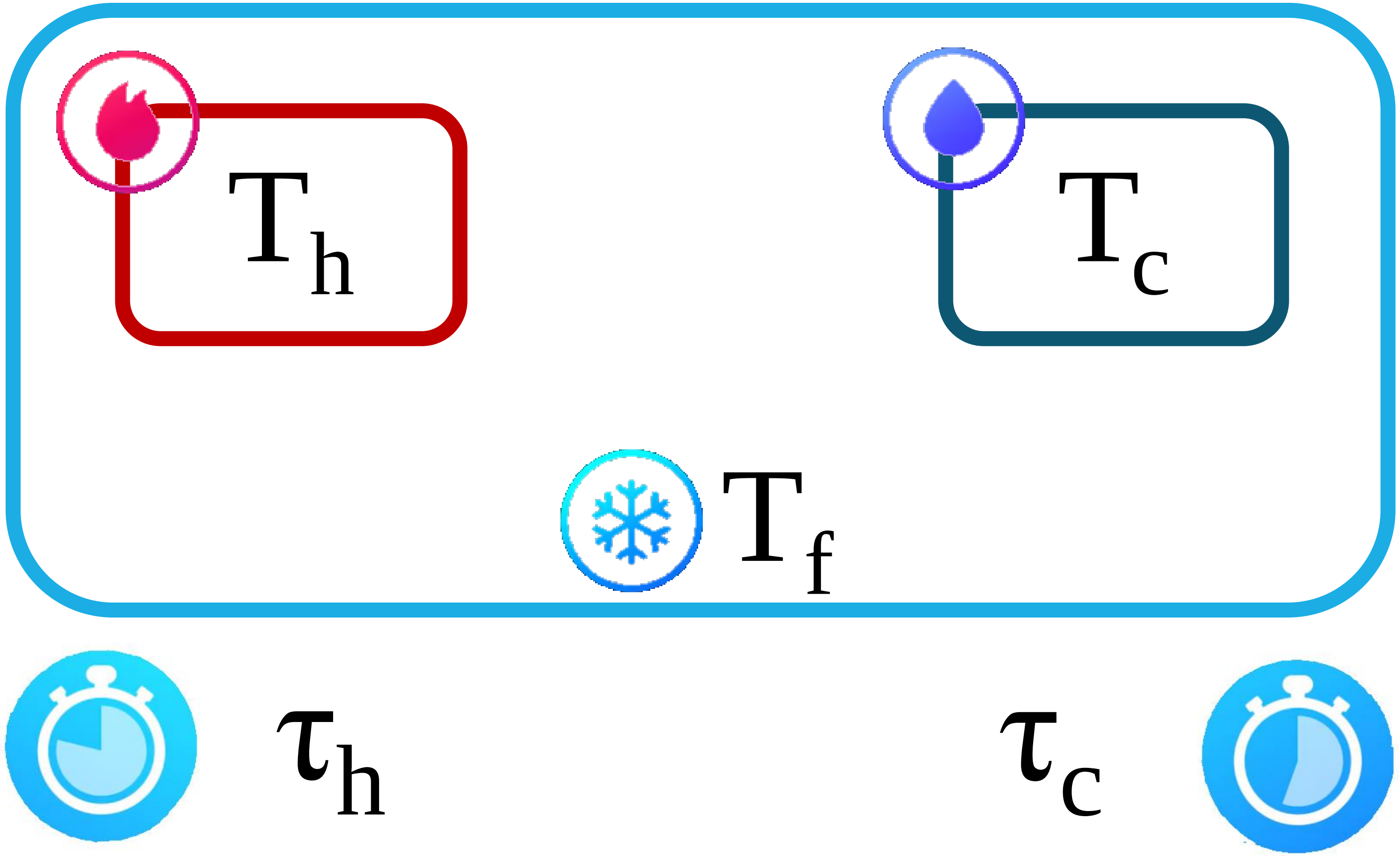


- Connect the systems to a thermal bath at subzero temperature

$$T_f < T_c < T_h$$

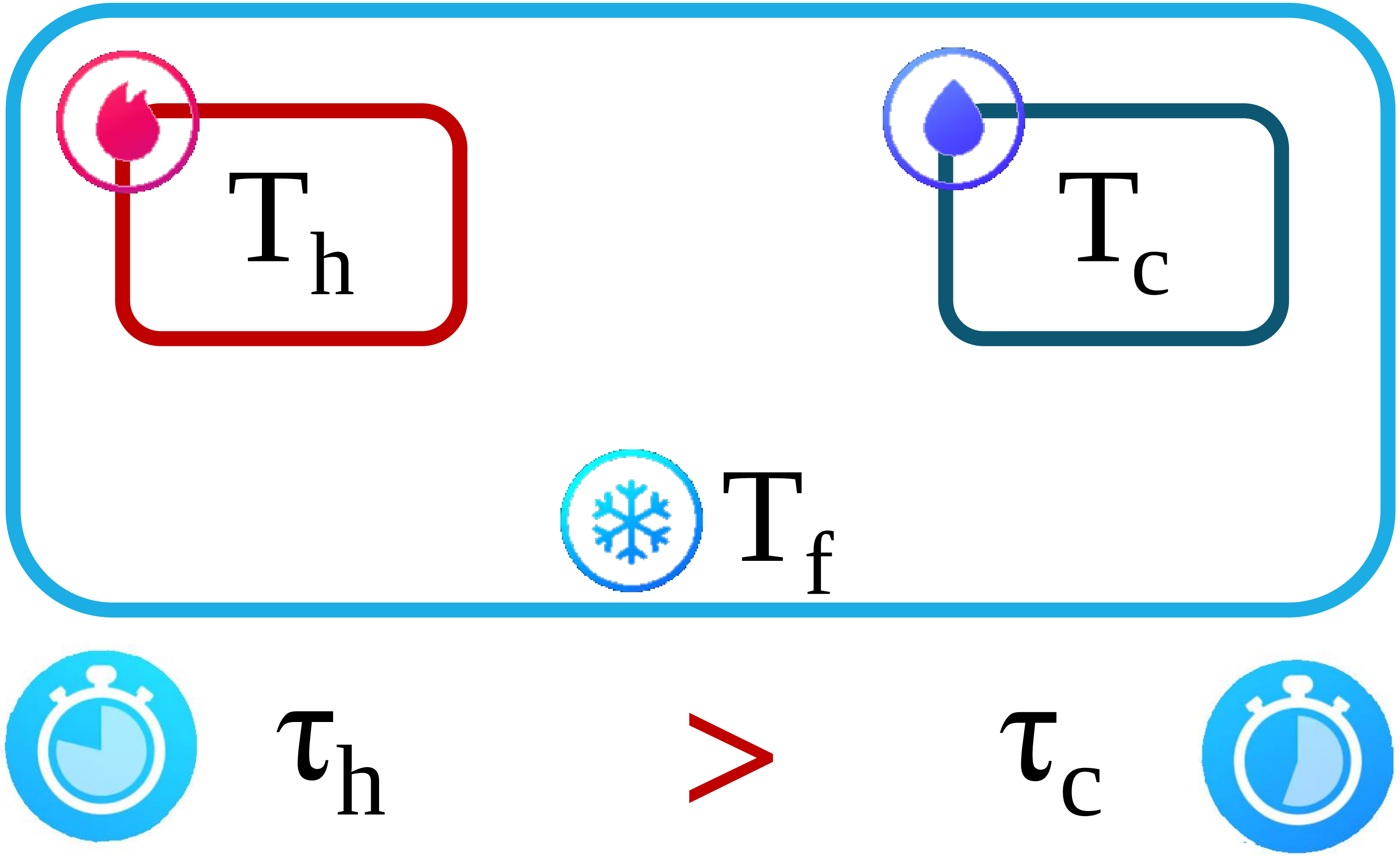


- Compute the relaxation time needed to freeze the systems

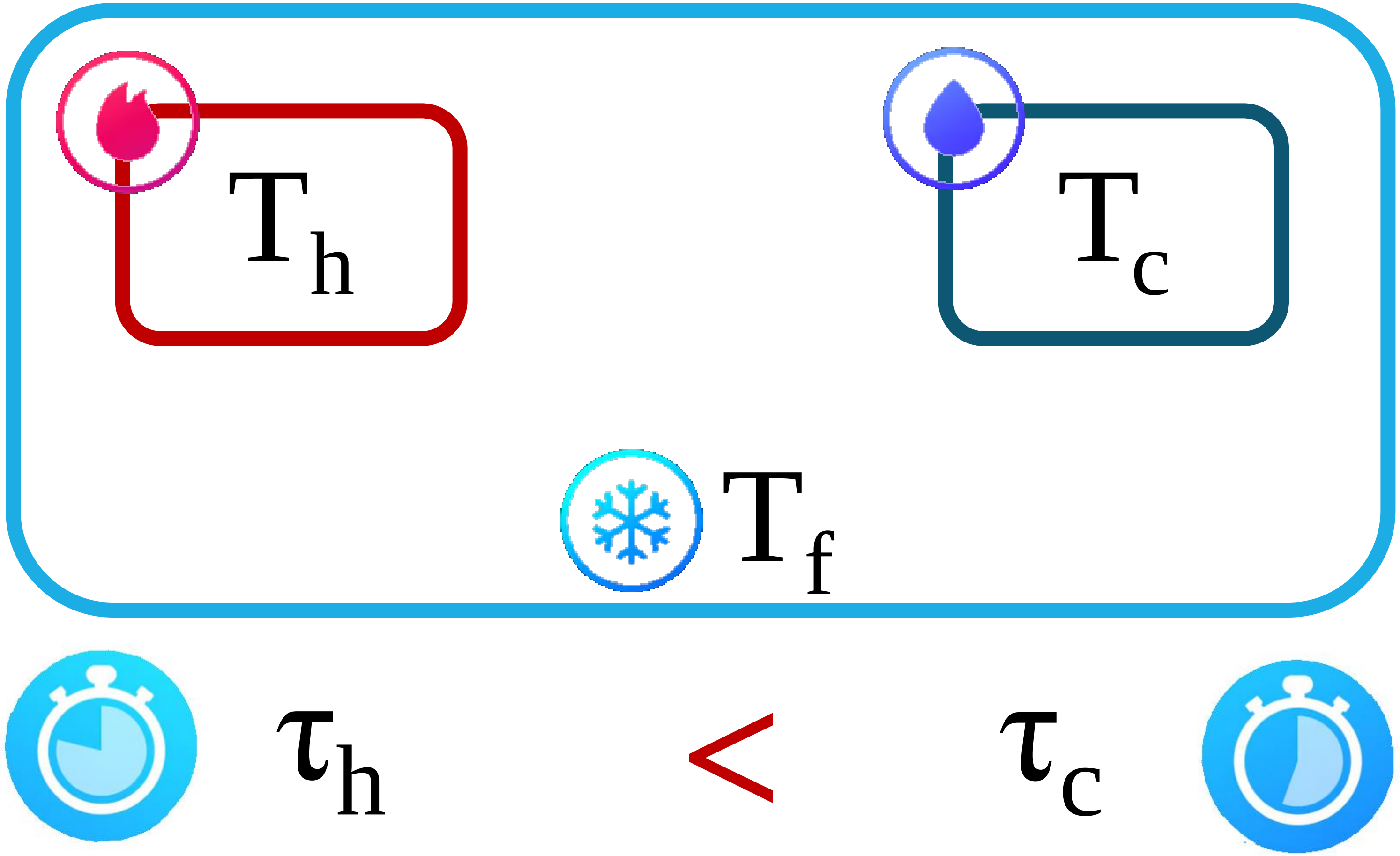


- <u>Newton heat law</u>: cold system freezes faster

$$T(t) = T_f + \left(T_0 - T_f\right)e^{-\frac{t}{\tau}}$$



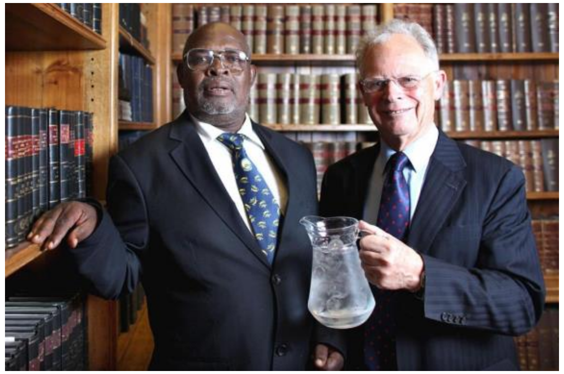
- ~~Newton heat law: cold system freezes faster~~
- Mpemba effect: hot system freezes faster

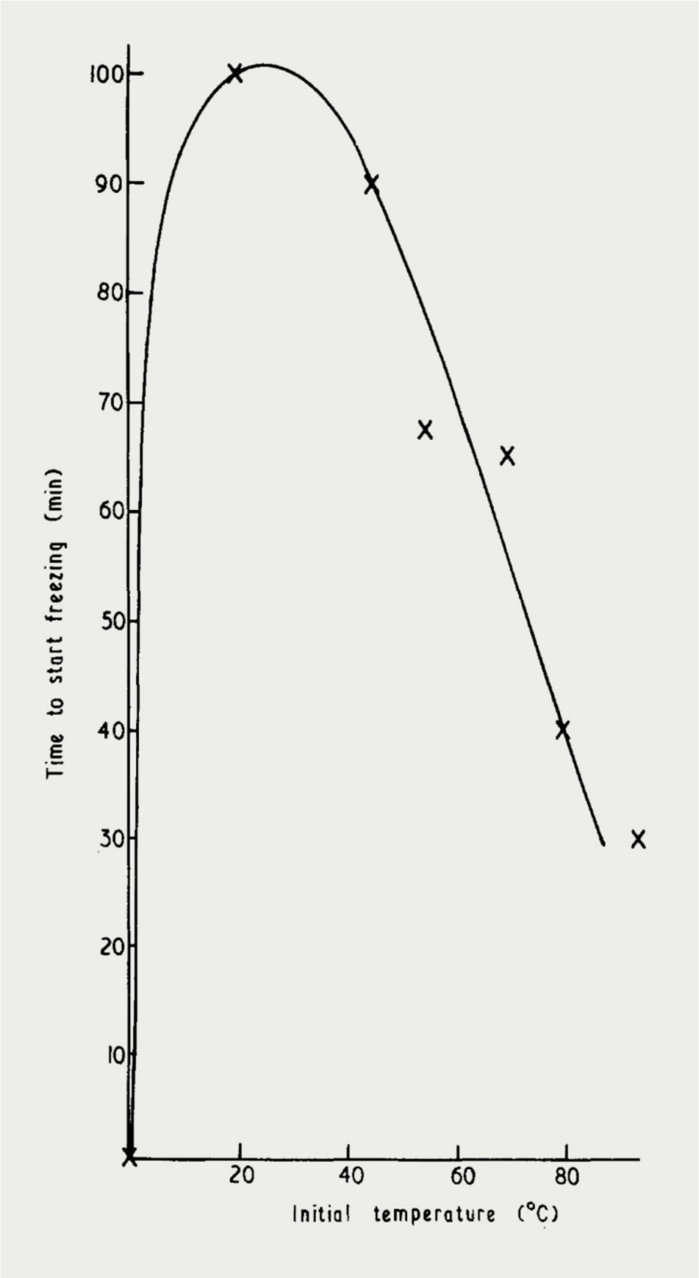


- 300 B.C. - Aristotle: “The fact that the water has previously been warmed contributes to its freezing quickly”

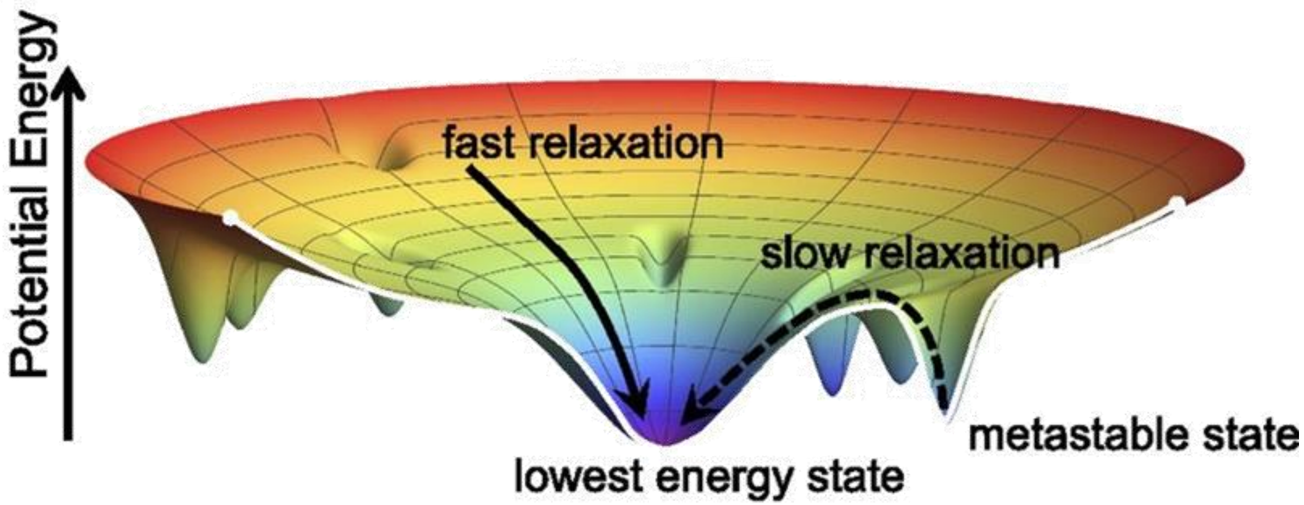
- 1461 - Giovanni Marliani, 1620 - Francis Bacon, 1637 - René Descartes

- 1963 - Erasto Mpemba (13-year old): he put hot ice-cream mix into the freezer, it froze more quickly than the cooled ice cream mix of his fellow classmates. E.B. Mpemba and D.G. Osborne, Cool? Phys. Educ., 1979. 14: 410-413

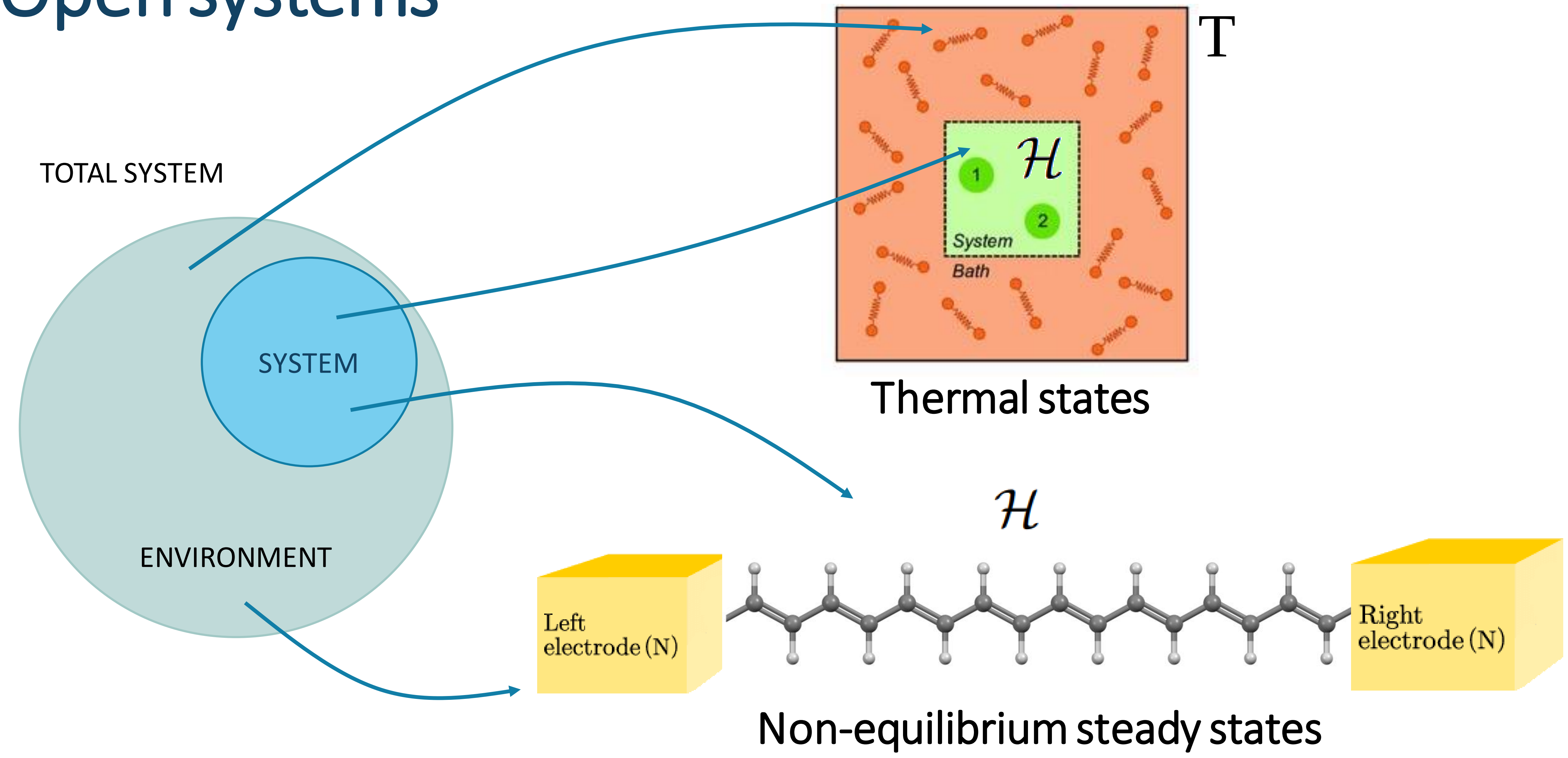




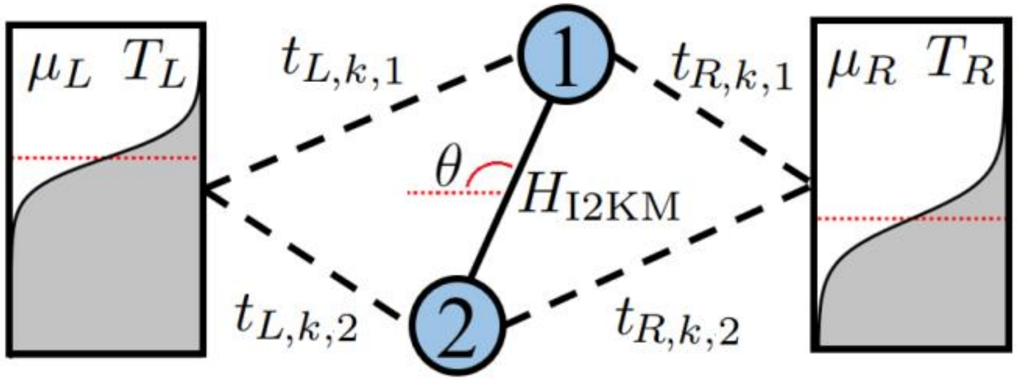
- <u>Different possible explanations</u> (Evaporation, Convection, Frost, Supercooling, Dissolved Gases) and <u>Experimentally observed</u> (Clathrate Hydrates, Magnetoresistance Alloys, Granular systems, Spin Glasses, Closed and Open Quantum systems)
- <u>Henry Burridge and Paul Linden</u>: the effect is sensible to the particulars of measurement and that the effect can exists also in **absence of a phase transition**
- <u>Zhiyue Lu and Oren Raz</u>: "We all have this naive picture that says temperature should change monotonically. You start at a high temperature, then a medium temperature, and go to a low temperature." But for nonequilibrium systems "it's not really true to say that the system has a temperature, since that's the case you can have **strange shortcuts**."



Open systems



- We investigate the *Quantum Mpemba Effect* for open nonequilibrium quantum systems coupled to (at least) two different reservoirs ("baths")



$$\{p\}$$

$$\rho(0)$$

- We introduce a general protocol to unambiguously identify the QME in open nonequilibrium systems with *Markovian* dynamics

$$\partial_t \rho = -i\left[H_{\mathrm{I2KM}}, \rho\right] + \sum_{m \neq n} \Gamma_{m,n} \mathcal{L}\left[L_{m,n}\right] \rho$$

$$\mathcal{L}[L]\rho \;=\; L\rho L^\dagger - \tfrac{1}{2}\{L^\dagger L, \rho\}$$

- **We need:**
    - a “master equation” for the system dynamics describing the evolution from the initial state to the (N)ESS
    - a “distance function” in parameter space that generalize the concept of “temperature quench”
    - a “distance function” in the density matrix space (state space) to describe how far the system is from the (N)ESS
    - a “velocity field” landscape to predict the presence of shortcuts

- We consider two sets of pre-quench bath/system parameters. At time $t = 0$, one performs a quench to the same after-quench parameters describing the final (N)ESS configuration

$$
\begin{array}{lll}
(\rho_{\mathrm{c}})\ \{p_{i.c}\} & \searrow & \\
\text{vs} & & \{p_{i,\mathrm{(N)ESS}}\}\ \ (\rho_{\mathrm{(N)ESS}}) \\
(\rho_{\mathrm{f}})\ \{p_{i,f}\} & \nearrow &
\end{array}
\qquad\qquad
\begin{array}{lll}
\mathrm{T_c} & \searrow & \\
 & & \mathrm{T_f} \\
\mathrm{T_h} & \nearrow &
\end{array}
$$

- We next extend the notion of "hot" vs "cold" initial configurations to "far" vs "close" initial conditions, to describe more general cases where several control parameters are changed. We introduce a distance function in the parameter space

$$\mathcal{D}_E\left(\{p_i\}\right) = \sqrt{\sum_i \left(p_i - p_{i,\mathrm{(N)ESS}}\right)^2} \qquad\qquad \mathcal{D}_E(\{p_{i,c}\}) < \mathcal{D}_E(\{p_{i,f}\})$$

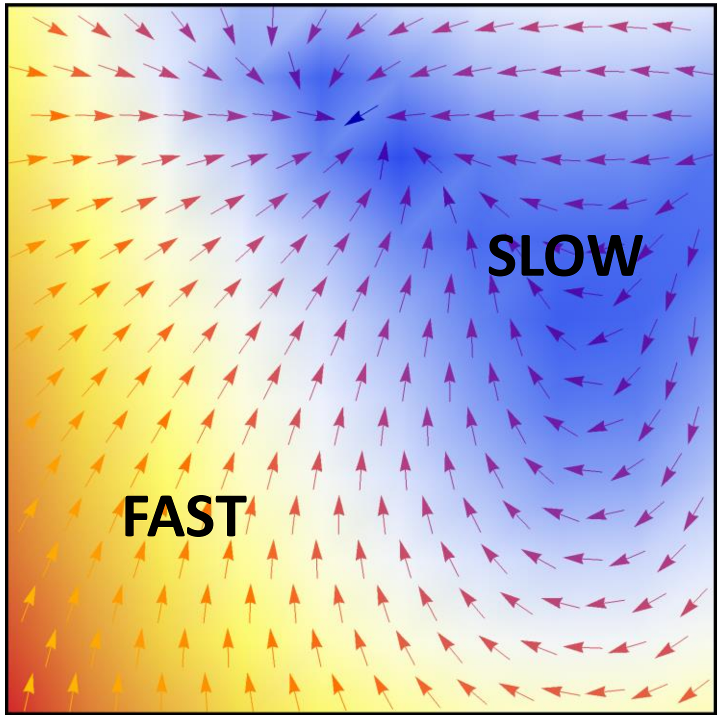
- We introduce the trace distance (monotonically nonincreasing, continuous, bounded, convex) to measure the distance between the system density matrix at time t and the final (N)ESS state

$$\mathcal{D}_T(\rho(t)) = \frac{1}{2}\mathrm{Tr}\left|\rho(t) - \rho_{\mathrm{(N)ESS}}\right|$$

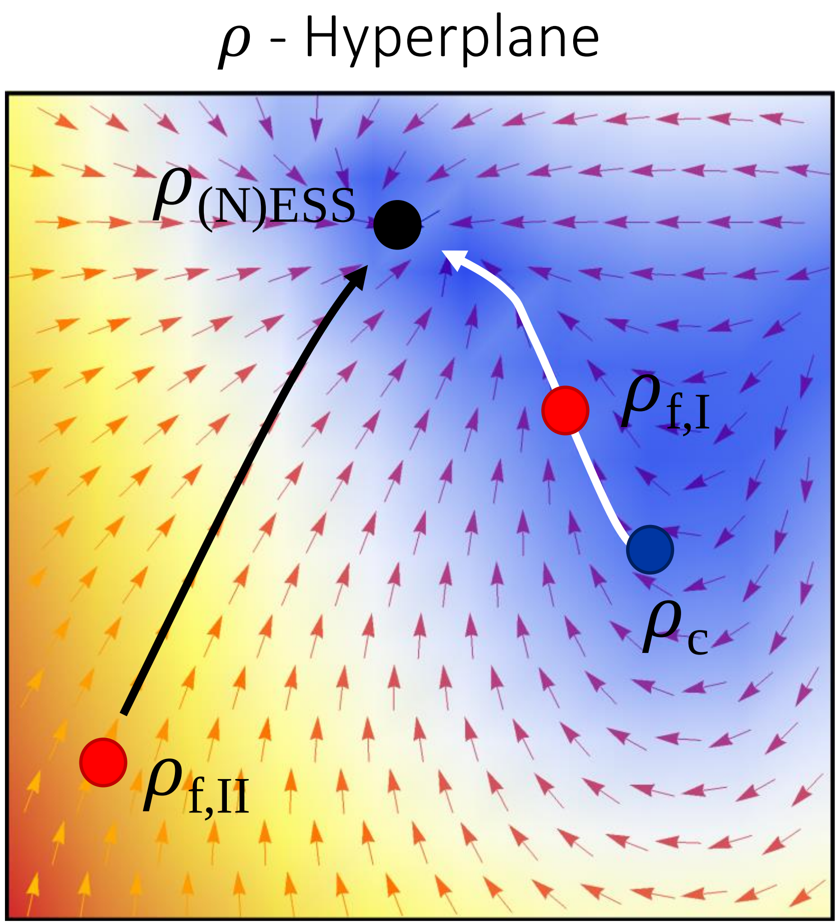
- We define the velocity field, which allows one to visualize in an intuitive manner if and how the Mpemba effect occurs.

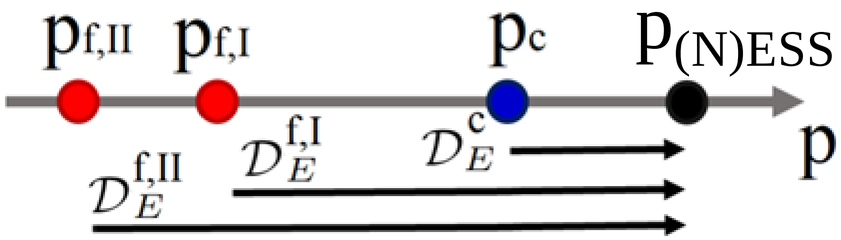
- The velocity field follows from the time derivative of the monitoring function

$$v(t) = \lim_{\delta t \to 0} \frac{\mathcal{D}_T\left(\rho(t+\delta t), \rho(t)\right)}{\delta t}$$



ρ - Hyperplane



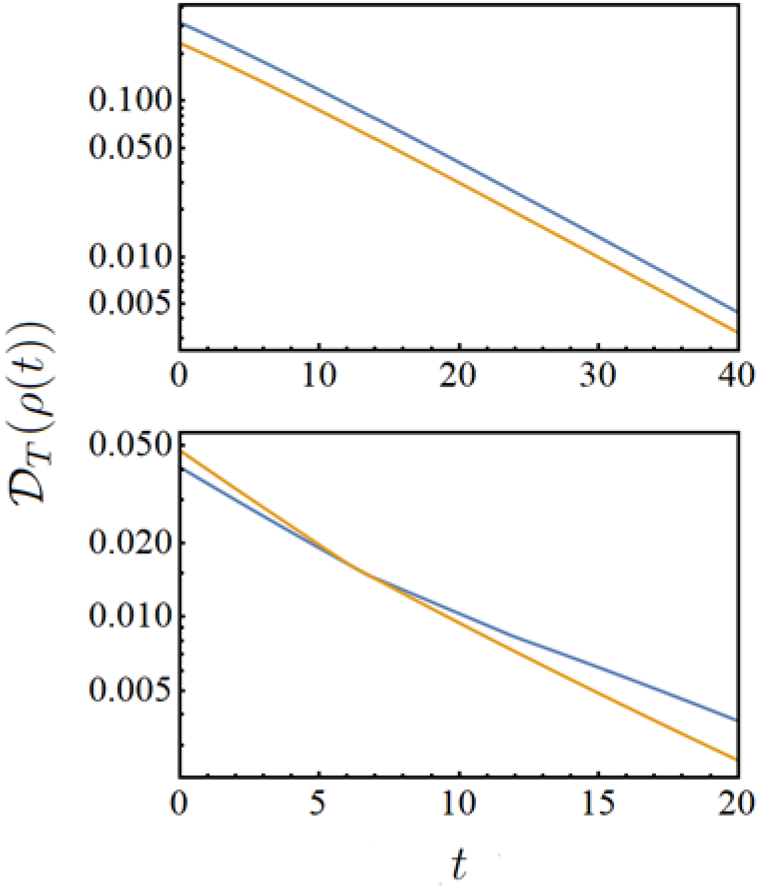


- For the type-I QME, the initially "far" system reaches the final (N)ESS faster ***<u>because</u>*** the corresponding density matrix is actually closer to the (N)ESS one

$$\mathcal{D}_T(\rho_f(0)) \; < \; \mathcal{D}_T(\rho_c(0))$$

- For the more elusive type-II QME, the "far" vs "close" systems evolve through different paths but ***<u>even though</u>*** the "closer" system is nearer to the(N)ESS, the total path of the "further" is eventually shorter

$$\mathcal{D}_T(\rho_f(0)) \; > \; \mathcal{D}_T(\rho_c(0))$$

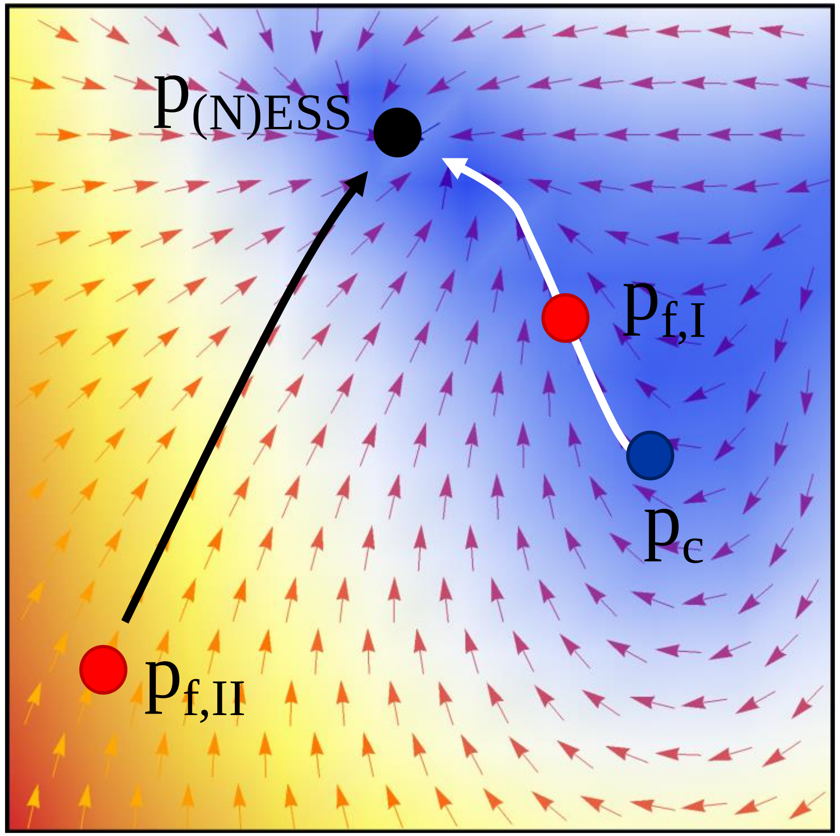


- Type-I QME

$$\mathcal{D}_T(\rho_f(t)) \;<\; \mathcal{D}_T(\rho_c(t))$$

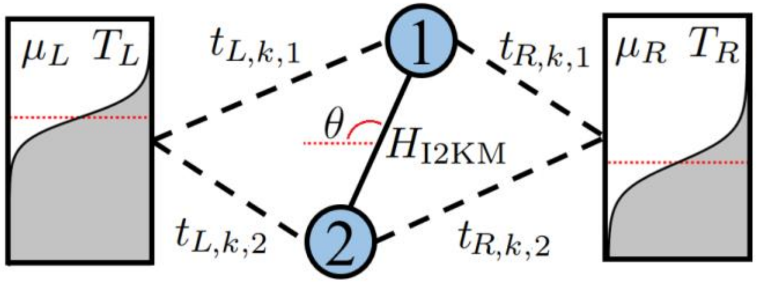
- Type-II QME

$$\mathcal{D}_T(\rho_f(0)) \;>\; \mathcal{D}_T(\rho_c(0))$$
$$\mathcal{D}_T(\rho_f(t)) \;<\; \mathcal{D}_T(\rho_c(t))$$



Example: Poor man's Kitaev model

- Minimal interacting two-site Kitaev model (Dvir et al. Nature volume 614, 445–450 (2023))



$$H_{\mathrm{I2KM}} = \epsilon_1 n_1 + \epsilon_2 n_2 + U n_1 n_2 + t_h(c_1^\dagger c_2 + c_2^\dagger c_1) + \Delta c_1^\dagger c_2^\dagger + \Delta^* c_2 c_1$$

- Environment: injection or removal of a single electron

$$L_{\mathrm{in},j} = \sqrt{\Gamma_j} c_j^\dagger \qquad L_{\mathrm{out},j} = \sqrt{\gamma_j} c_j$$

$$\Gamma_{i,0}^\lambda = \gamma_{\lambda,i}(\epsilon_i) f_\lambda(\epsilon_i)$$
$$\Gamma_{d,i}^\lambda = \gamma_{\lambda,3-i}(\epsilon_d - \epsilon_i) f_\lambda(\epsilon_d - \epsilon_i)$$

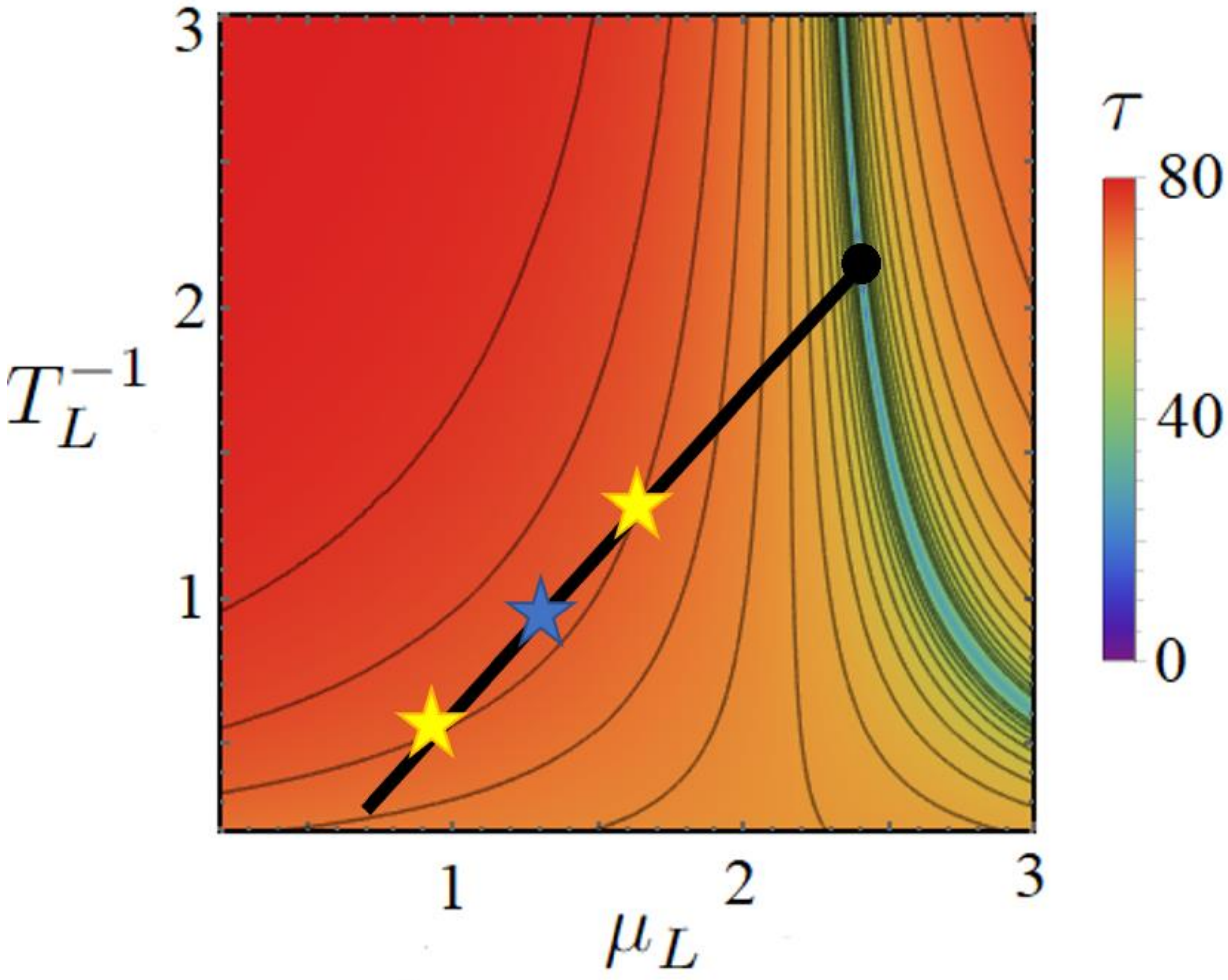
- The rates connect different parity sectors

$$\{|0\rangle, |d\rangle\} \leftrightarrow \{|1\rangle, |2\rangle\}$$

- Conversely, the Hamiltonian only connects states with same parity

$$|1\rangle \leftrightarrow |2\rangle \qquad |0\rangle \leftrightarrow |d\rangle$$

- Relaxation time from trace distance



Thin black lines: isolines of τ

Fixed parameters:

$t_h = 1, \epsilon_1 = \epsilon_2 = 2,$
$U = 0.25, \Delta = 0.1,$
$\Gamma = 0.1, \theta = 0,$
$T_R^{-1} = 2.15, \mu_R = 2.4$

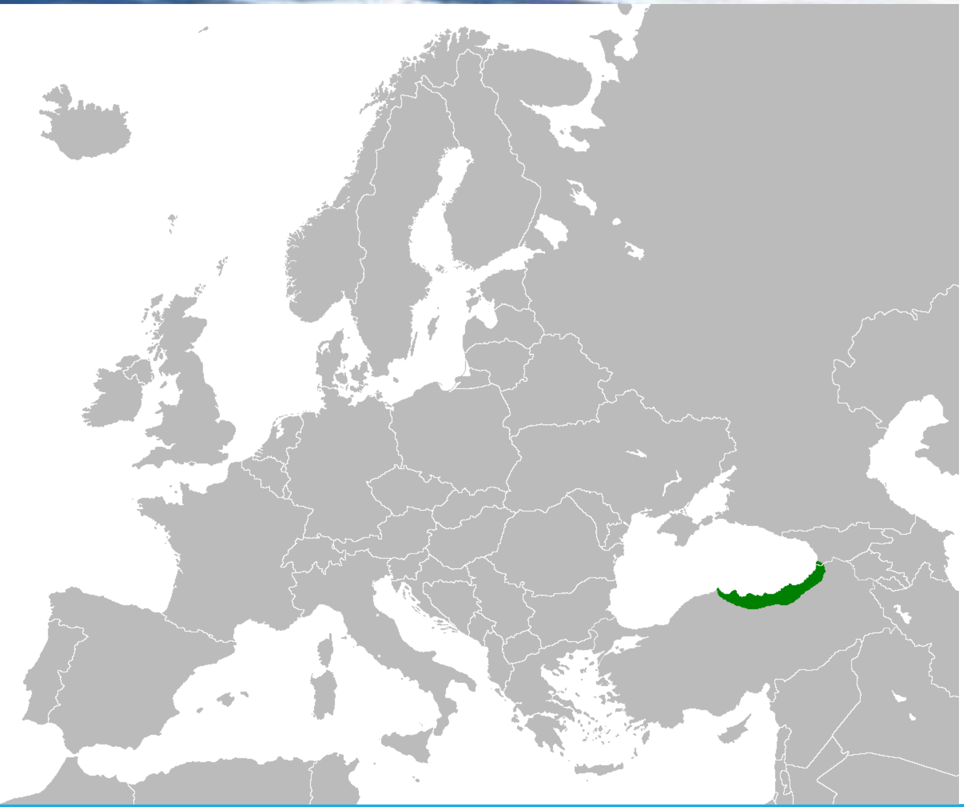
Quench parameters:

$\boldsymbol{T_L^{-1}, \mu_L}$

Towards:

$T_L^{-1} = 2.15, \mu_L = 2.4$

FISHING WITH ARISTOTLE



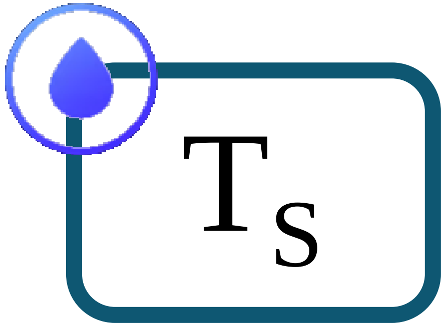
- <u>300 B.C. - Aristotle</u>: "The fact that the water has previously been warmed contributes to its freezing quickly: for so it cools sooner. Hence many people, when they want to cool water quickly, begin by putting it in the sun. So the inhabitants of Pontus when they encamp on the ice to fish (they cut a hole in the ice and then fish), pour warm water round their reeds that it may freeze the quicker, for they use the ice like lead to fix the reeds"

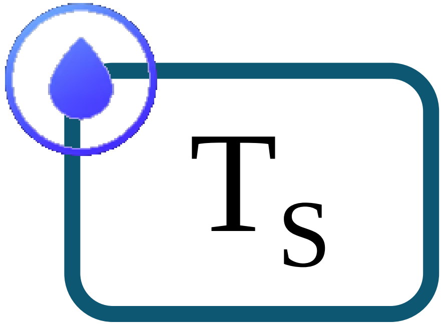
↓

Aristotle suggests a <u>two step</u> optimization protocol!

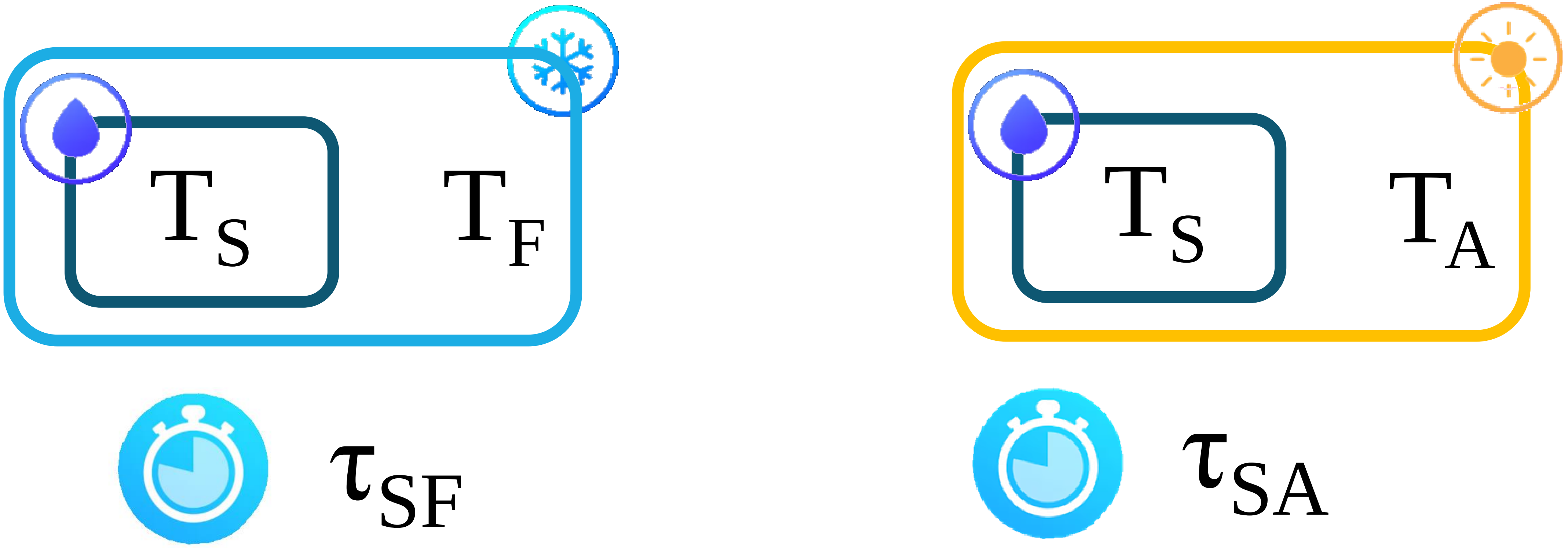
↓

Pontus-Mpemba effect

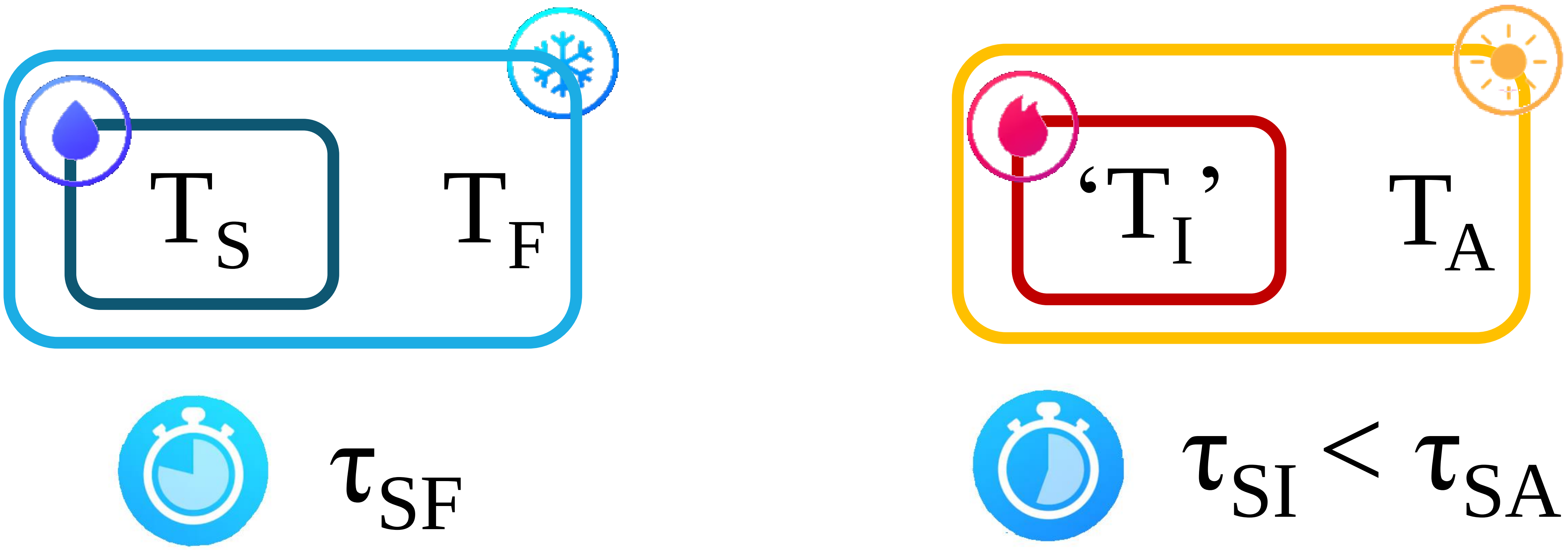




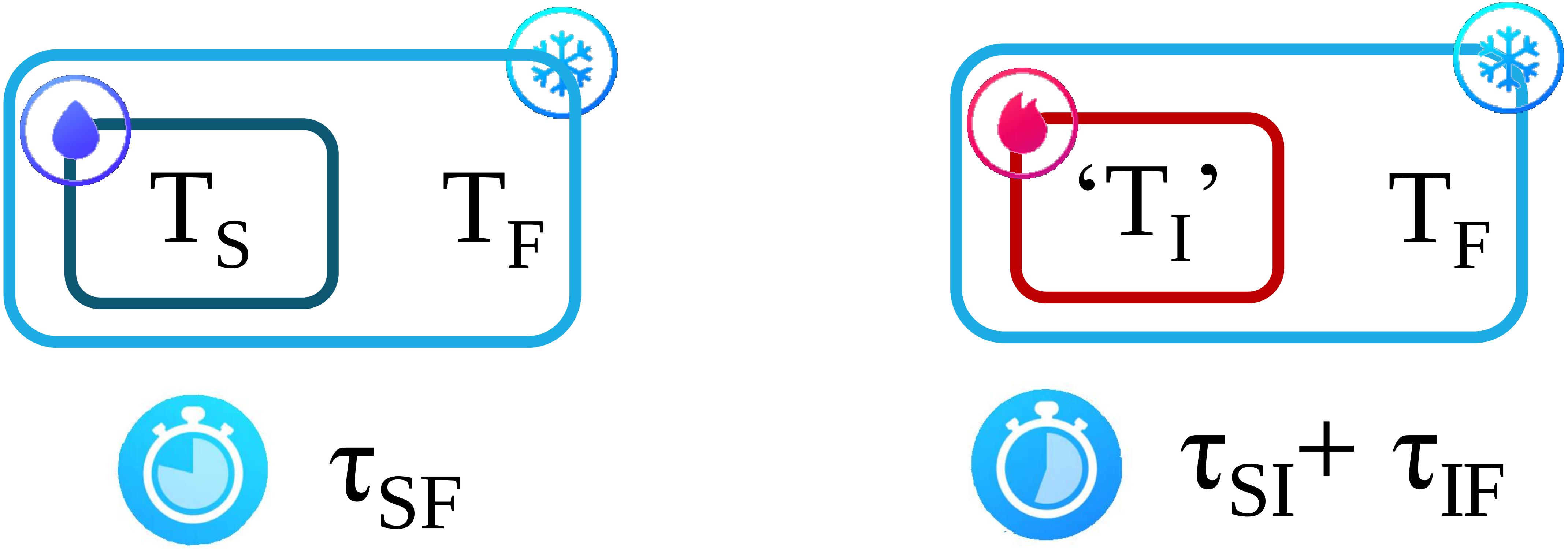
- We consider two system copies prepared in the same initial state S



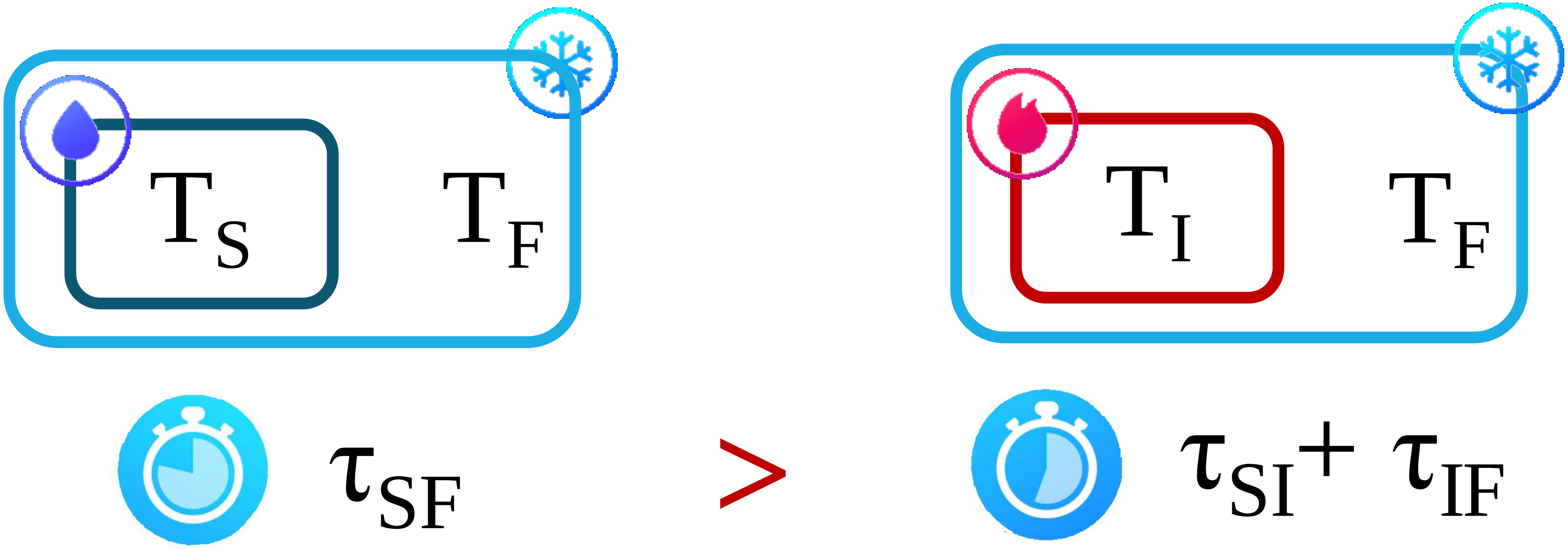
- At time t = 0, the first copy is subjected to a control parameter quench driving the system toward the target state F in a time span τ_{SF}
- The second copy is first subjected to a different parameter quench driving the system toward the stationary state A in a time span τ_{SA}



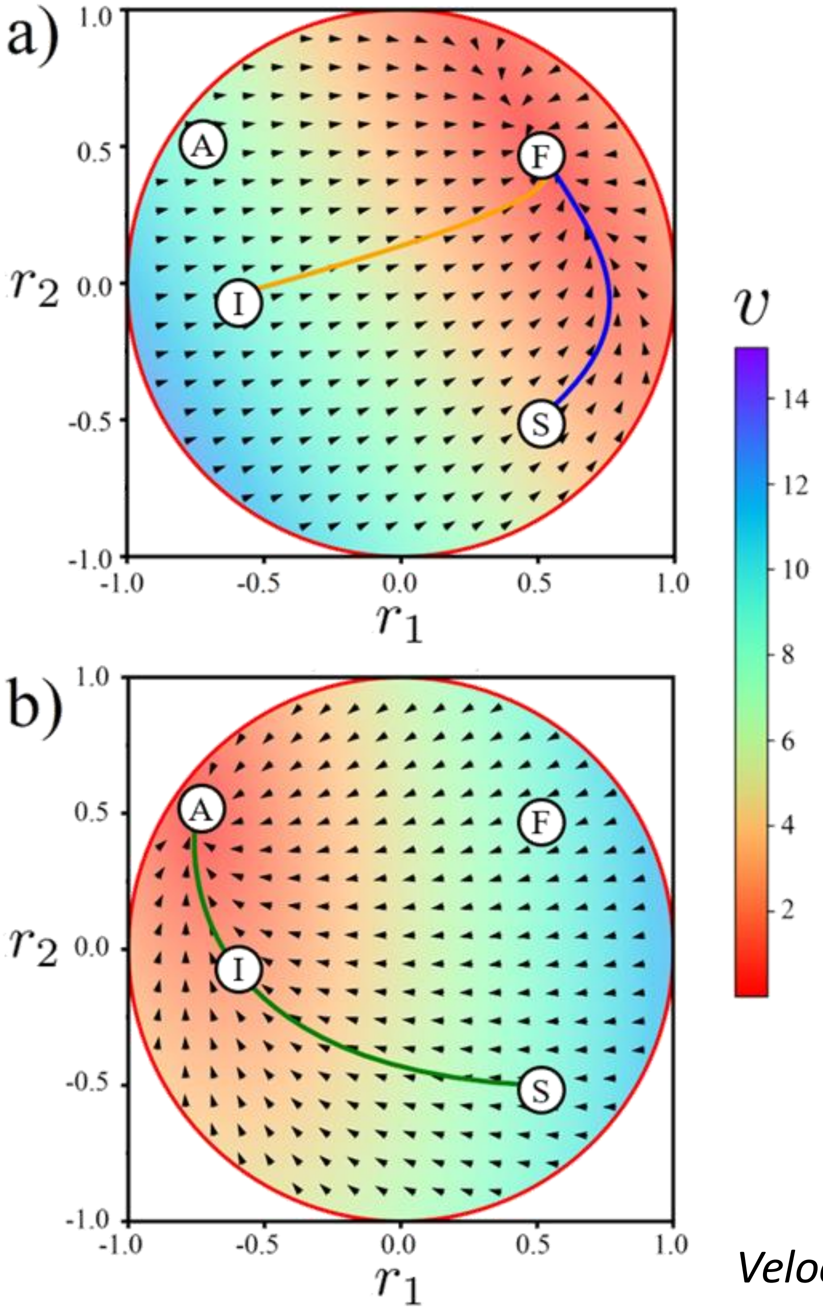
- During the evolution toward A, at a time $\tau_{SI} < \tau_{SA}$, where the intermediate state I has been reached, the system is decoupled from the auxiliary environment



- By a second parameter quench, the system is then connected to the same environment used for the first system copy. Then reaches the target state F in a time τ_{IF}



- The Pontus-Mpemba effect occurs if the condition $\tau_{SI} + \tau_{IF} < \tau_{SF}$ is satisfied



Velocity field

- In the above example by Aristotle, S corresponds to the cold water, F to ice, A is induced by the sun (or by fire) and would eventually lead to evaporated water, and I is the warm water. While for the standard Mpemba effect, I is a stationary (and usually thermal) state, for the Pontus-Mpemba effect, there is no such constraint. Indeed, I is now just a generic state along the time evolution from S to A

- The relative position between the trajectories brings to a classification of possible Pontus-Mpemba effects types:

  - weak type-A

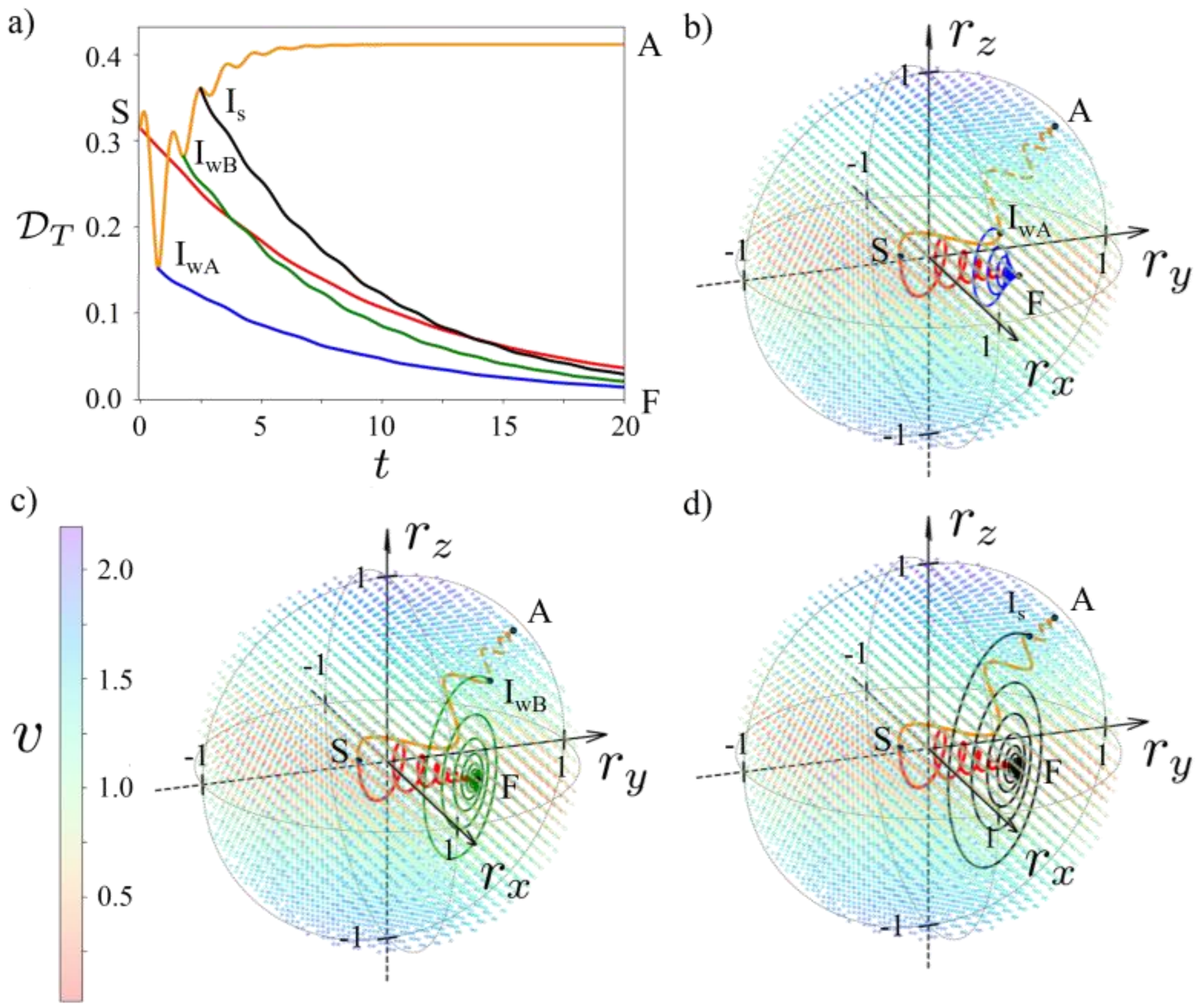
$$\mathcal{D}_T(\rho^{(\mathrm{A})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}}) < \mathcal{D}_T(\rho^{(\mathrm{F})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}})$$

  - weak type-B

$$\mathcal{D}_T(\rho_{\mathrm{S}}, \rho_{\mathrm{F}}) > \mathcal{D}_T(\rho^{(\mathrm{A})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}}) > \mathcal{D}_T(\rho^{(\mathrm{F})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}})$$

  - strong

$$\mathcal{D}_T(\rho^{(\mathrm{A})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}}) > \mathcal{D}_T(\rho_{\mathrm{S}}, \rho_{\mathrm{F}})$$



- Pontus-Mpemba idea can be further generalized by considering not just a single additional quench but a sequence of parameter variations

- In the limit of infinitely many infinitesimal quenches, the protocol becomes continuous, and the generator of the dynamics becomes explicitly time-dependent

$$\frac{d\rho(t)}{dt} = \mathcal{L}[\rho(t)] = -i\,[H(t),\rho(t)] + \sum_{\lambda} \gamma_\lambda(t) \left(L_\lambda(t)\rho(t)L_\lambda^\dagger(t) - \frac{1}{2}\left\{ L_\lambda^\dagger(t)L_\lambda(t), \rho(t) \right\} \right)$$

- The <u>continuous</u> Pontus-Mpemba protocol interpolates between two limiting regimes: the <u>quasi-static</u> limit (the parameters vary slowly compared to the relaxation rate) and the <u>sudden-quench</u> Mpemba limit (where parameters are changed abruptly)

- Non-Markovian features arise if the decay rates change sign during the evolution

Conclusions

- We have formulated a general and widely applicable protocol which allows one to unambiguously identify and classify the QME in open nonequilibrium quantum systems

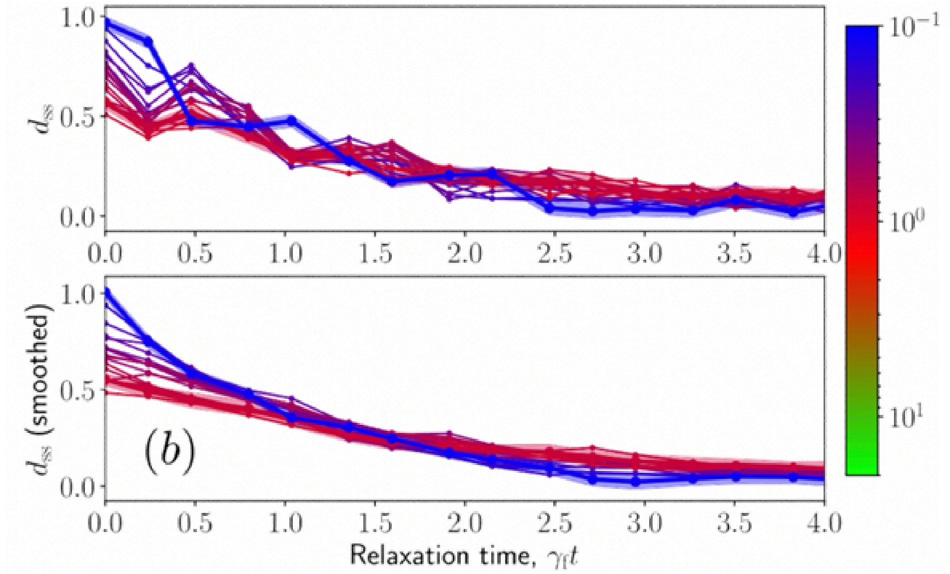
- We have introduced the Pontus-Mpemba effect as a generalized protocol realizing Mpemba physics by taking into account the time needed to prepare the initially “far” state. This protocol is more demanding but also more stringent than the standard Mpemba protocol.

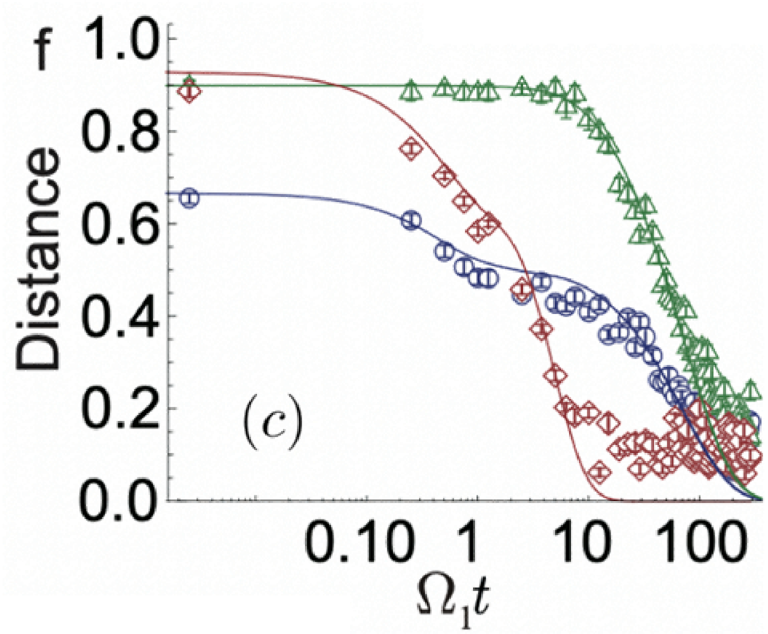
- Many open questions remains. From fundamental theoretical inquiries to potential technological applications!

THANK YOU

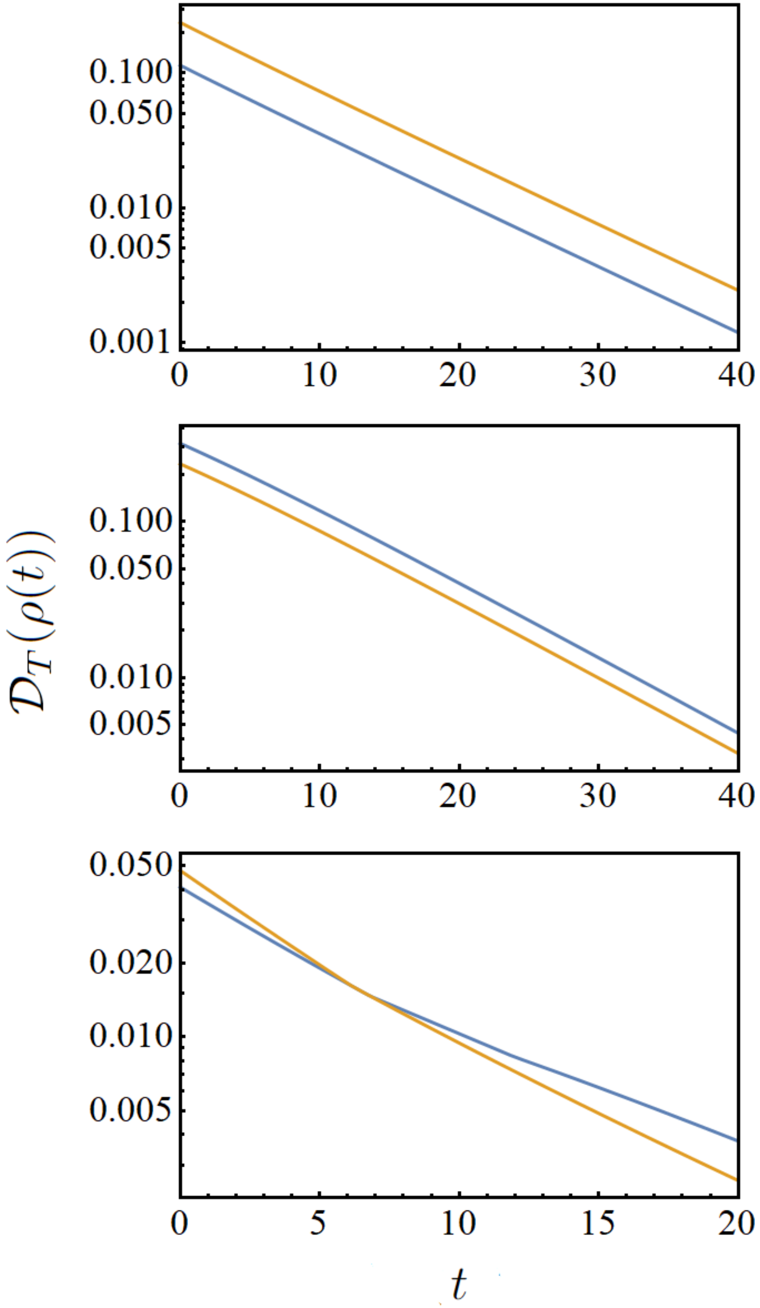
- In the last few years the attention moved from classical macroscopic systems to the quantum realm (less controversial, possibility to understand and predict the phenomena)

S. A. Shapira, O. Raz, R. Ozeri et al.
Inverse Mpemba Effect Demonstrated on a Single Trapped Ion Qubit,
Phys. Rev. Lett. 133, 010403 (2024)

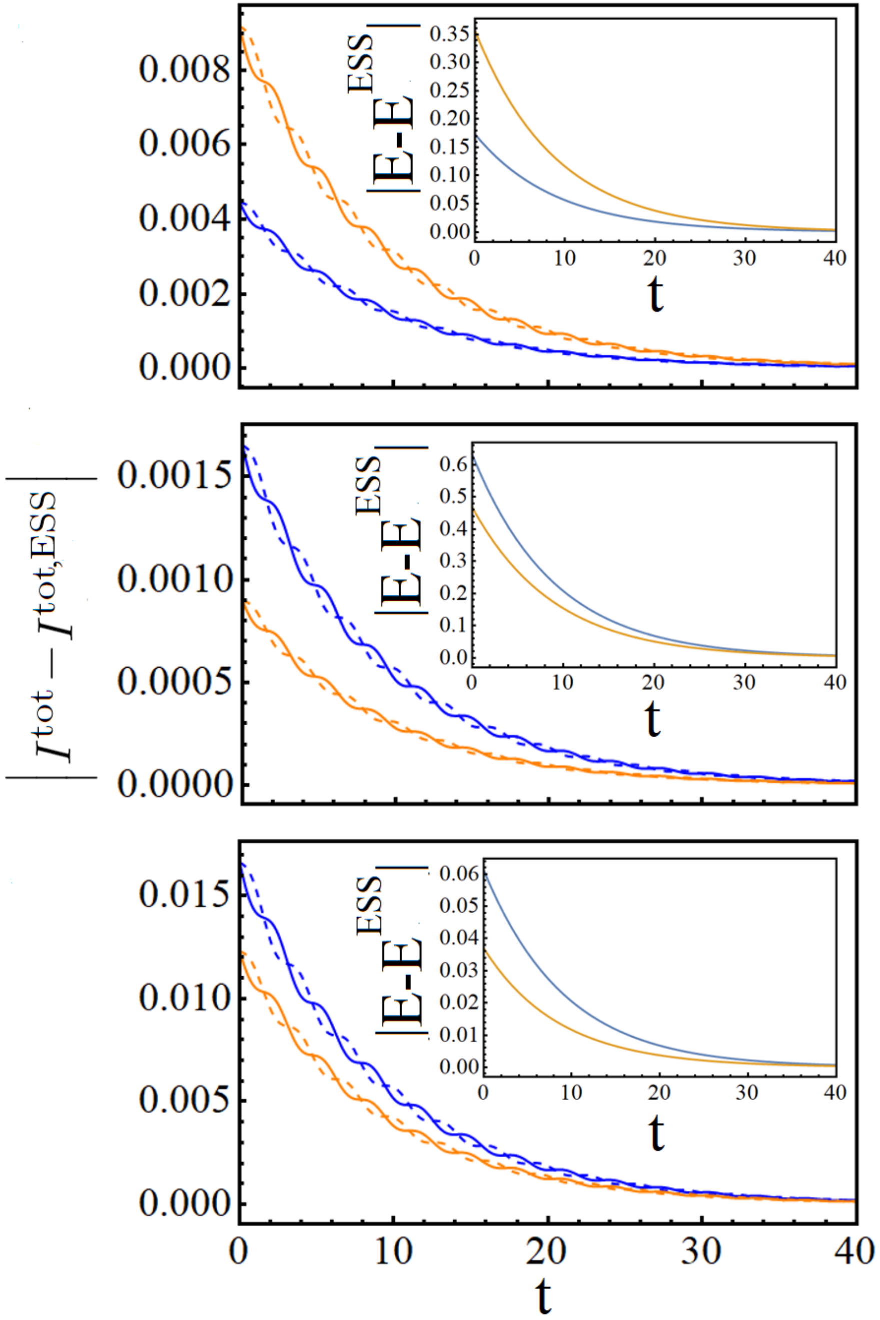




J. Zhang, Y.-L. Zhou et al.
Observation of quantum strong Mpemba effect,
Nature Comms. 16, 301 (2025)



- No QME
- Type-I QME
- Type-II QME



- Trace distance to monitor the systems evolution

$$\mathcal{D}_T(\rho_1, \rho_2) = \frac{1}{2}\mathrm{Tr}\,|\rho_1 - \rho_2|$$

- The relative position between the trajectories brings to a classification of possible Pontus-Mpemba effects types:
  - weak type-I

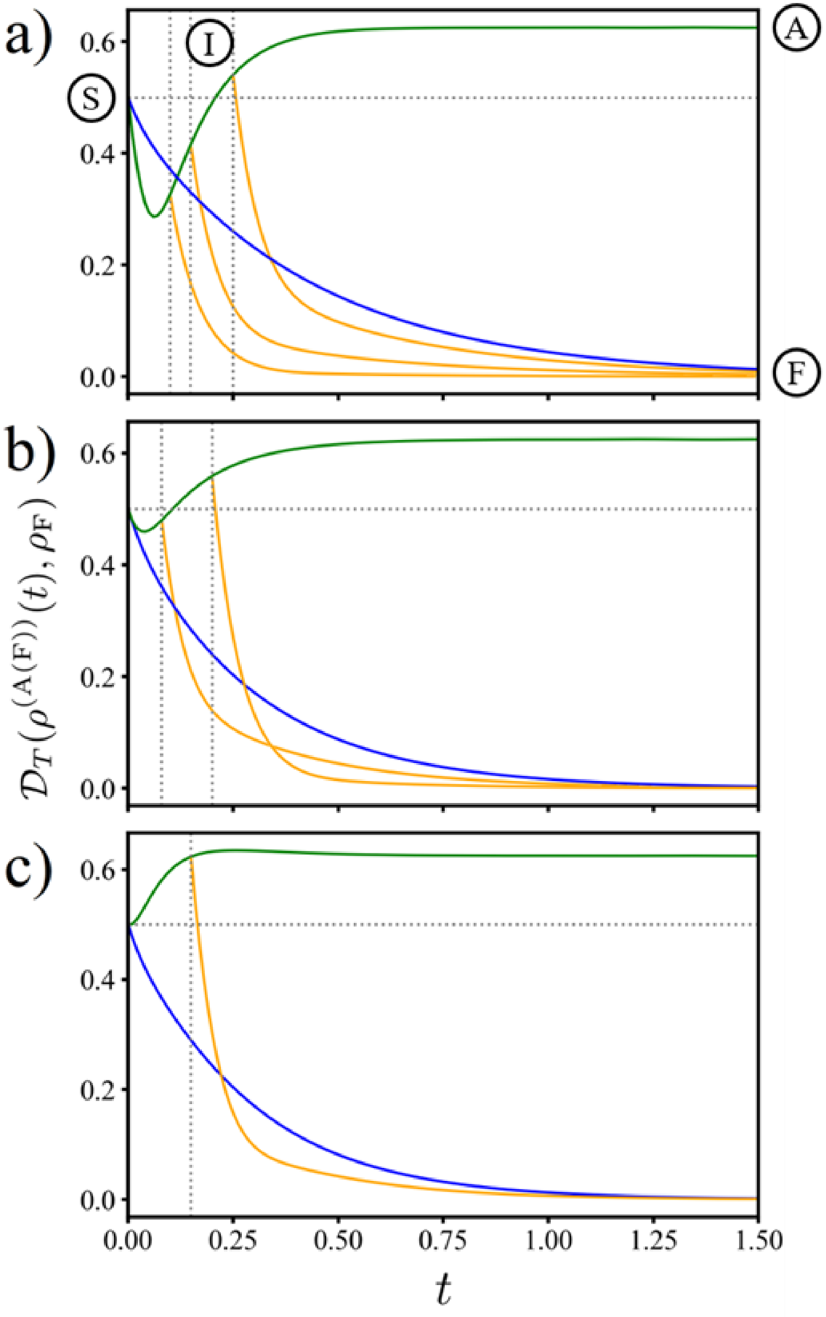
$$\mathcal{D}_T(\rho^{(\mathrm{A})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}}) < \mathcal{D}_T(\rho^{(\mathrm{F})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}})$$

  - weak type-II

$$\mathcal{D}_T(\rho_{\mathrm{S}}, \rho_{\mathrm{F}}) > \mathcal{D}_T(\rho^{(\mathrm{A})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}}) > \mathcal{D}_T(\rho^{(\mathrm{F})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}})$$

  - strong

$$\mathcal{D}_T(\rho^{(\mathrm{A})}(t_{\mathrm{SI}}), \rho_{\mathrm{F}}) > \mathcal{D}_T(\rho_{\mathrm{S}}, \rho_{\mathrm{F}})$$

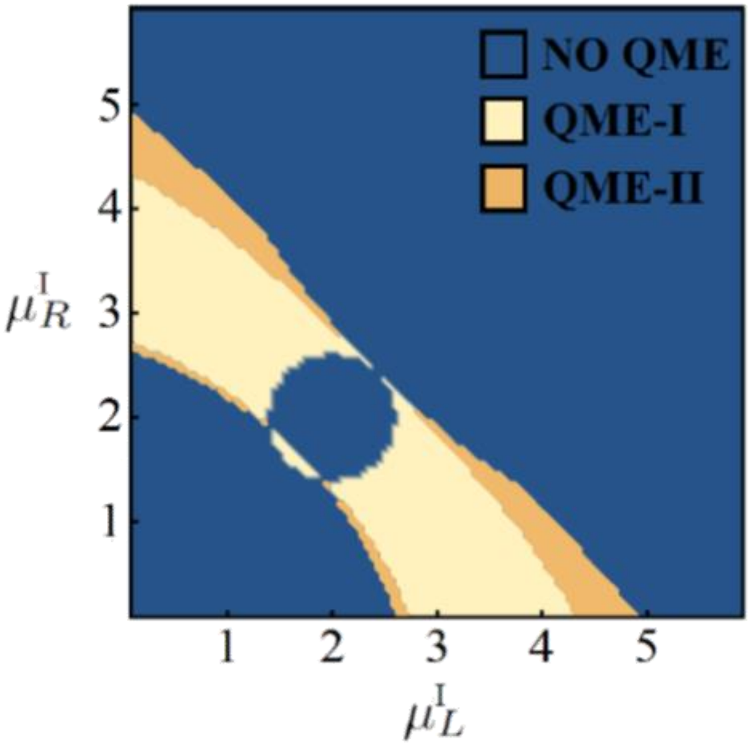


Example 2: Anderson dot

- Consider spinful interacting single-level dot coupled to metallic leads by tunneling:

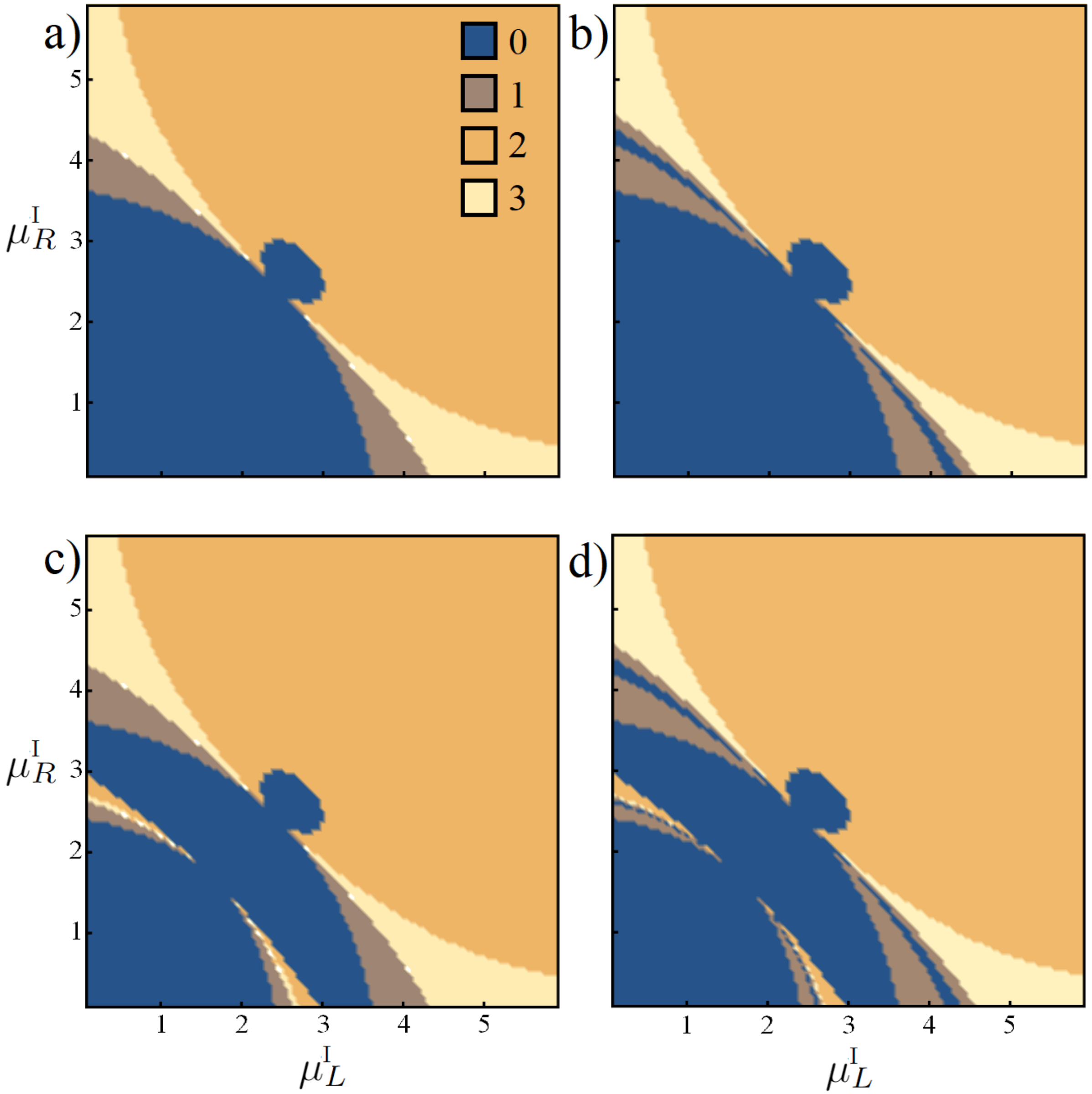
$$H_0 = \epsilon(n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow$$

- Jump operators in Lindblad equation now mix all states → equations for diagonal elements of density matrix ("populations") decouple → Pauli master equation



Phase diagram for $\Gamma = 1, U = 1.25, \epsilon = 2$
final ESS $kT = 1, \mu_L = \mu_R = 2,$
and reference initial state
$kT_L^{II} = kT_R^{II} = 0.85 , \mu_L^{II} = \mu_R^{II} = 2.43,$
using $kT_L^I = kT_R^I = 0.85$

Quench parameters: μ_L^I, μ_R^I

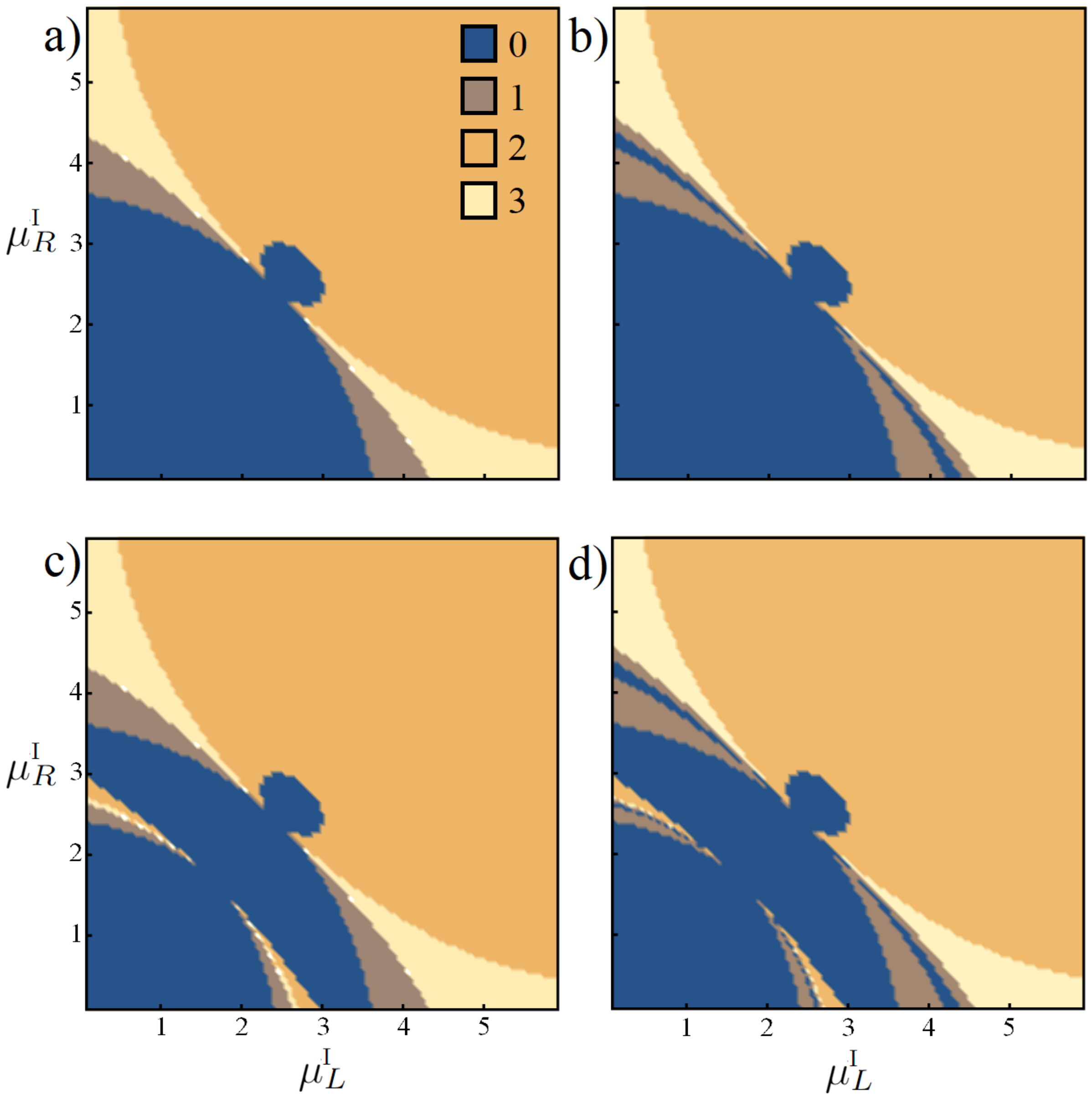


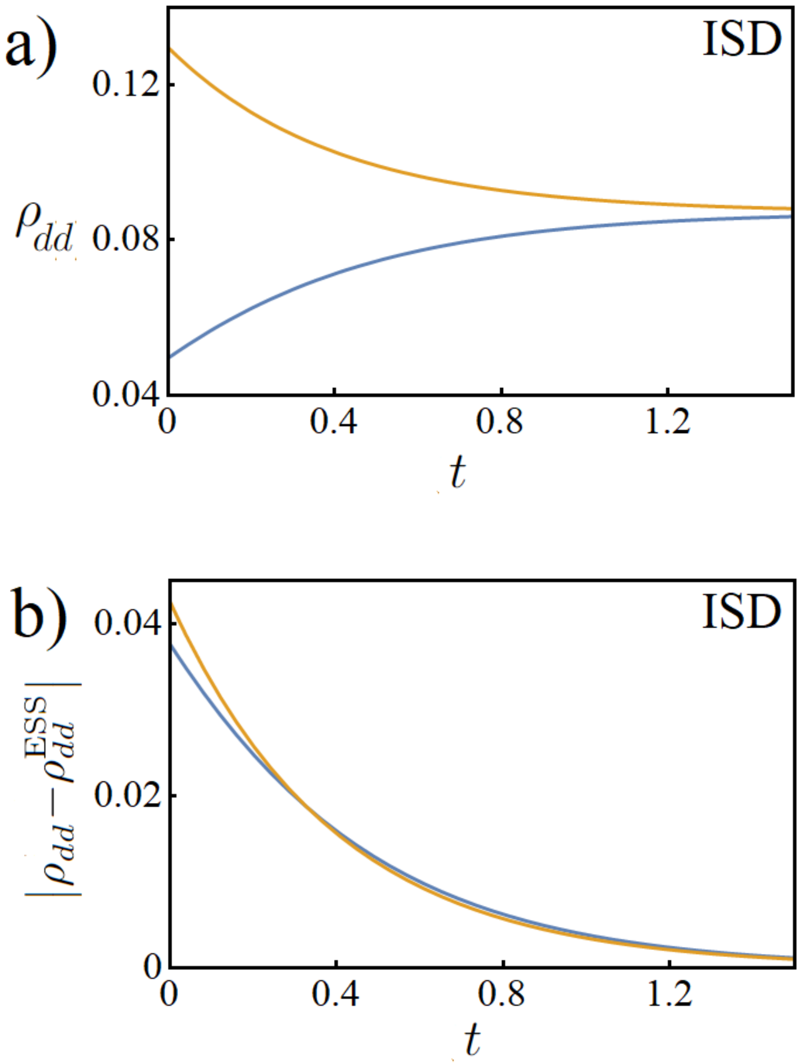
- Number of crossing points in the "population" time evolution

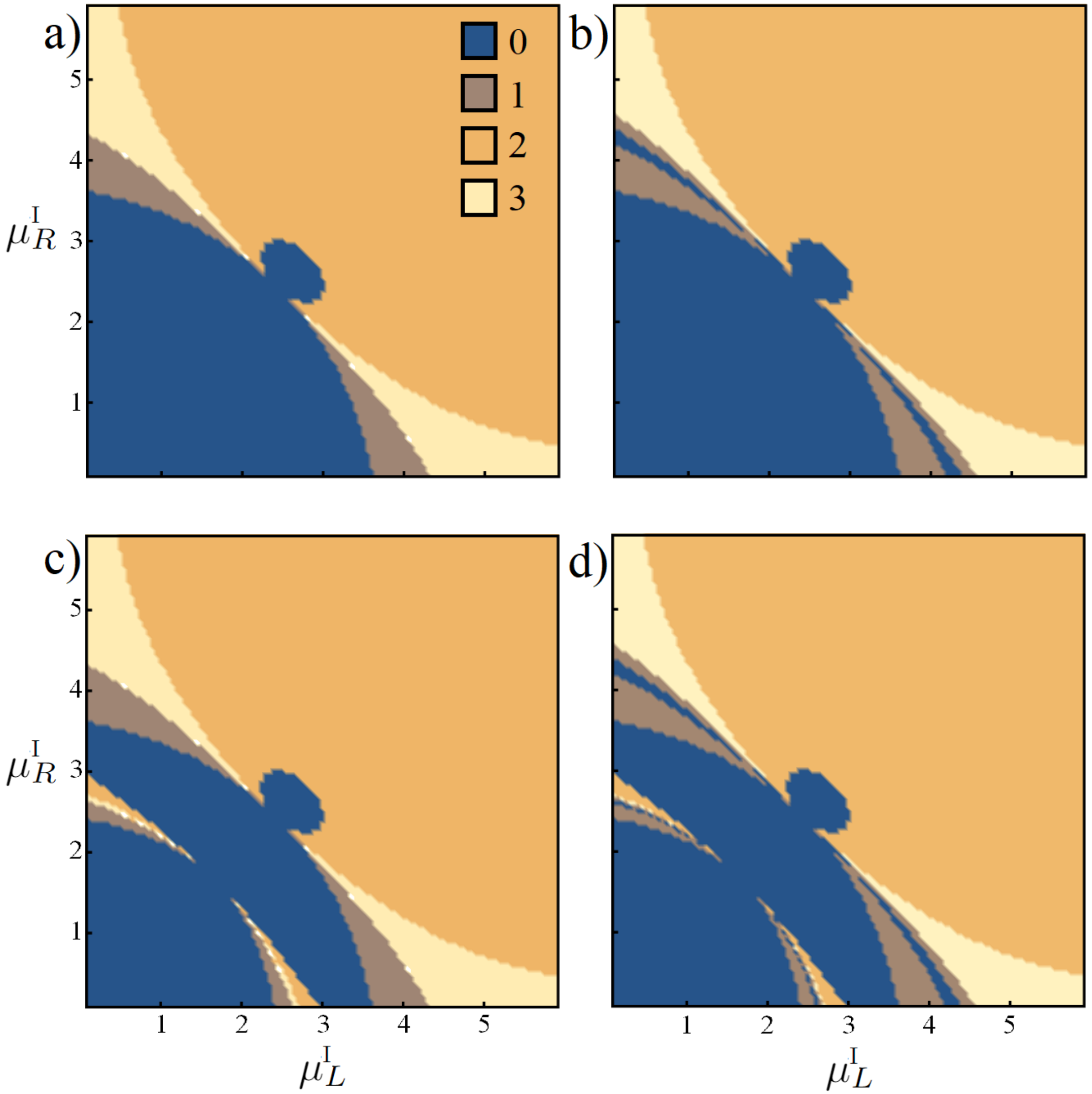
$$\mathbf{P}(t) \equiv \begin{pmatrix} \rho_{0,0}(t) \\ \rho_{\uparrow,\uparrow}(t) \\ \rho_{\downarrow,\downarrow}(t) \\ \rho_{d,d}(t) \end{pmatrix}$$

- However, it is neither a ***necessary*** nor a ***sufficient*** condition for the onset of the Mpemba Effect

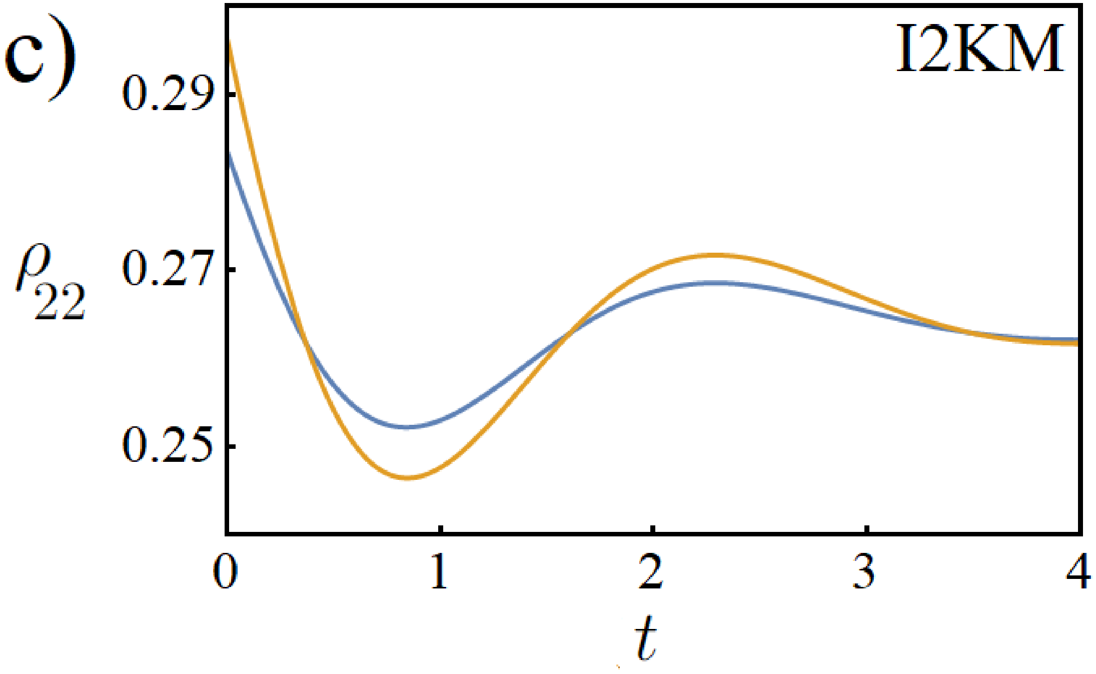
- Not *necessary*





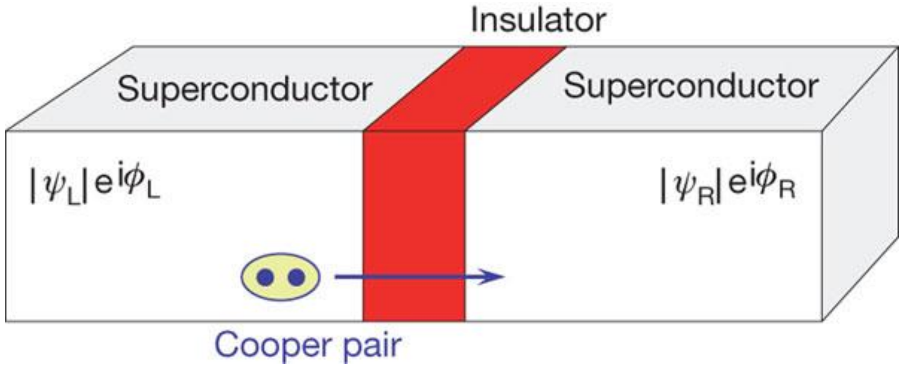


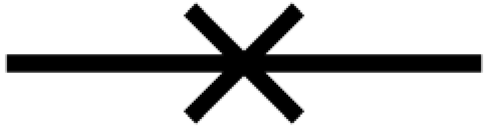
- Not *sufficient*



Example 3: Superconducting weak link

- A <u>Josephson junction</u> consists of two superconductors separated by a very thin insulating (or weakly conducting) barrier



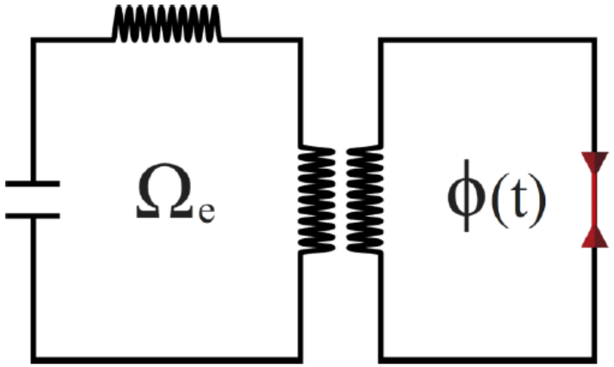


- One giant quantum wave describing all Cooper pairs. In superconductors, phase difference is a physical variable. A difference in phase directly generates current

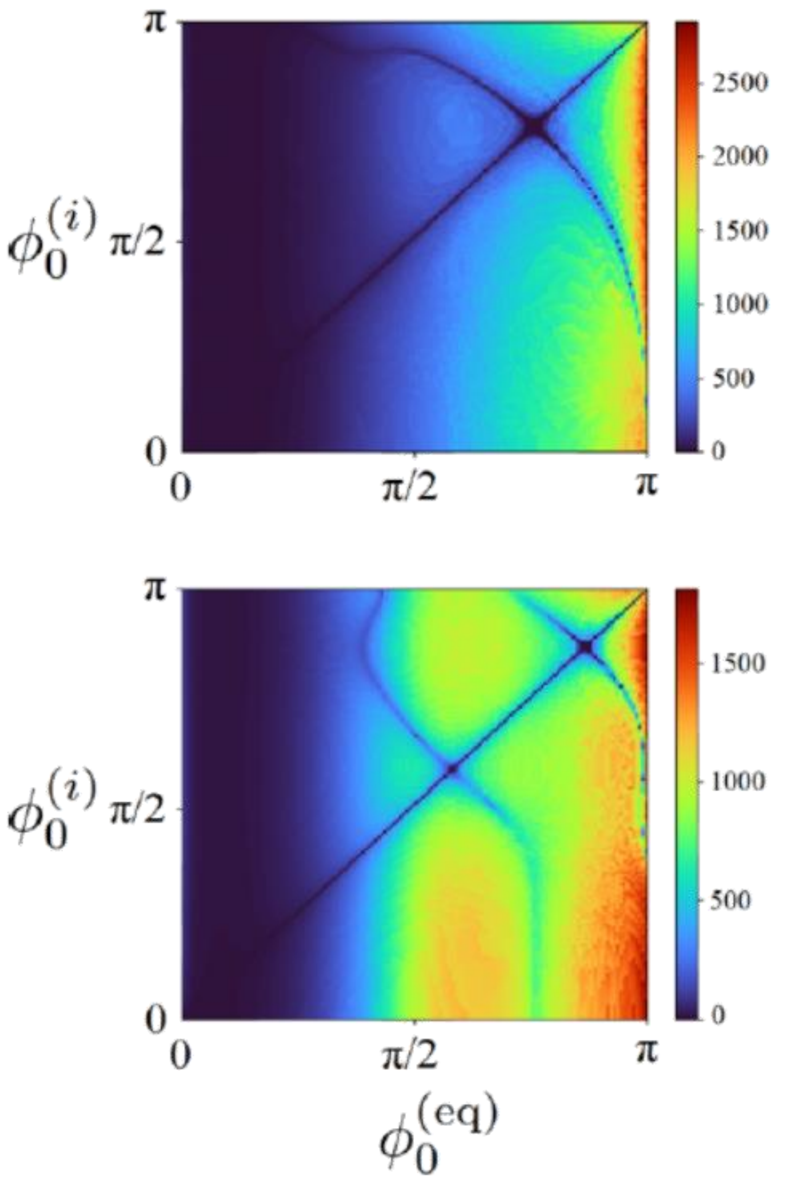
- A Josephson junction converts quantum phase differences into measurable electrical currents

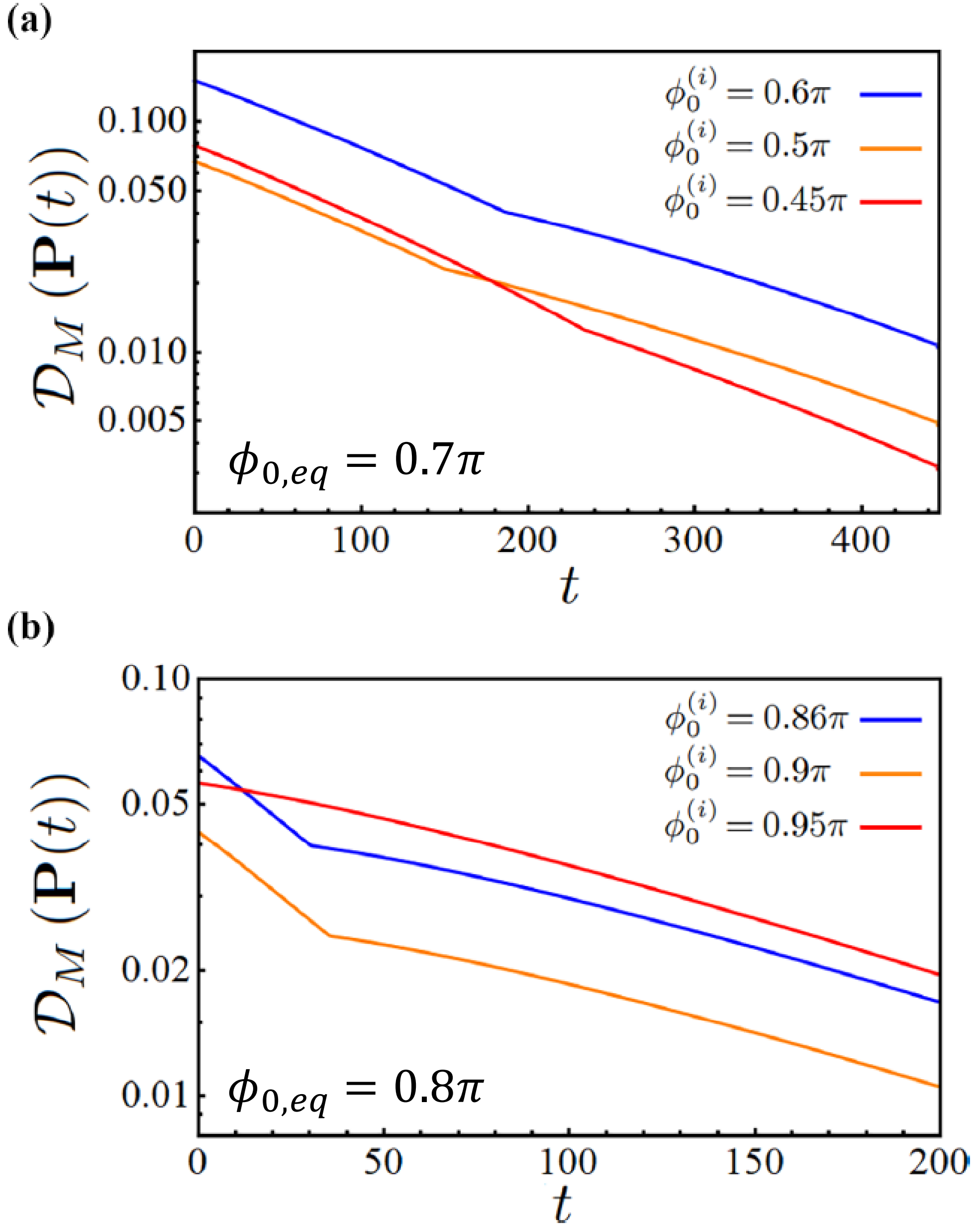
K. Zatsarynna, A.N., R. Egger & A. Zazunov, Phys. Rev. B 111, 104506 (2025)

- Josephson junction containing noninteracting quantum dot, embedded in microwave environment → QME for quench of superconducting phase difference across weak link



- Sharp features (open curves in initial-final phase plane) visible for relaxation time and trace distance: Mpemba arcs





- All three behaviors (no QME, type-I or type-II QME) are again possible, depending on parameters (in particular, junction transparency)
- Relaxation time accessible by microwave spectroscopy
- To distinguish type-I and type-II QME, time-dependent distance measure has to be monitored → Andreev level occupations

Example 4: Time-dependent dissipation

- The time-dependent dissipation rates can be modelled by an exponential decay together with an oscillatory modulation

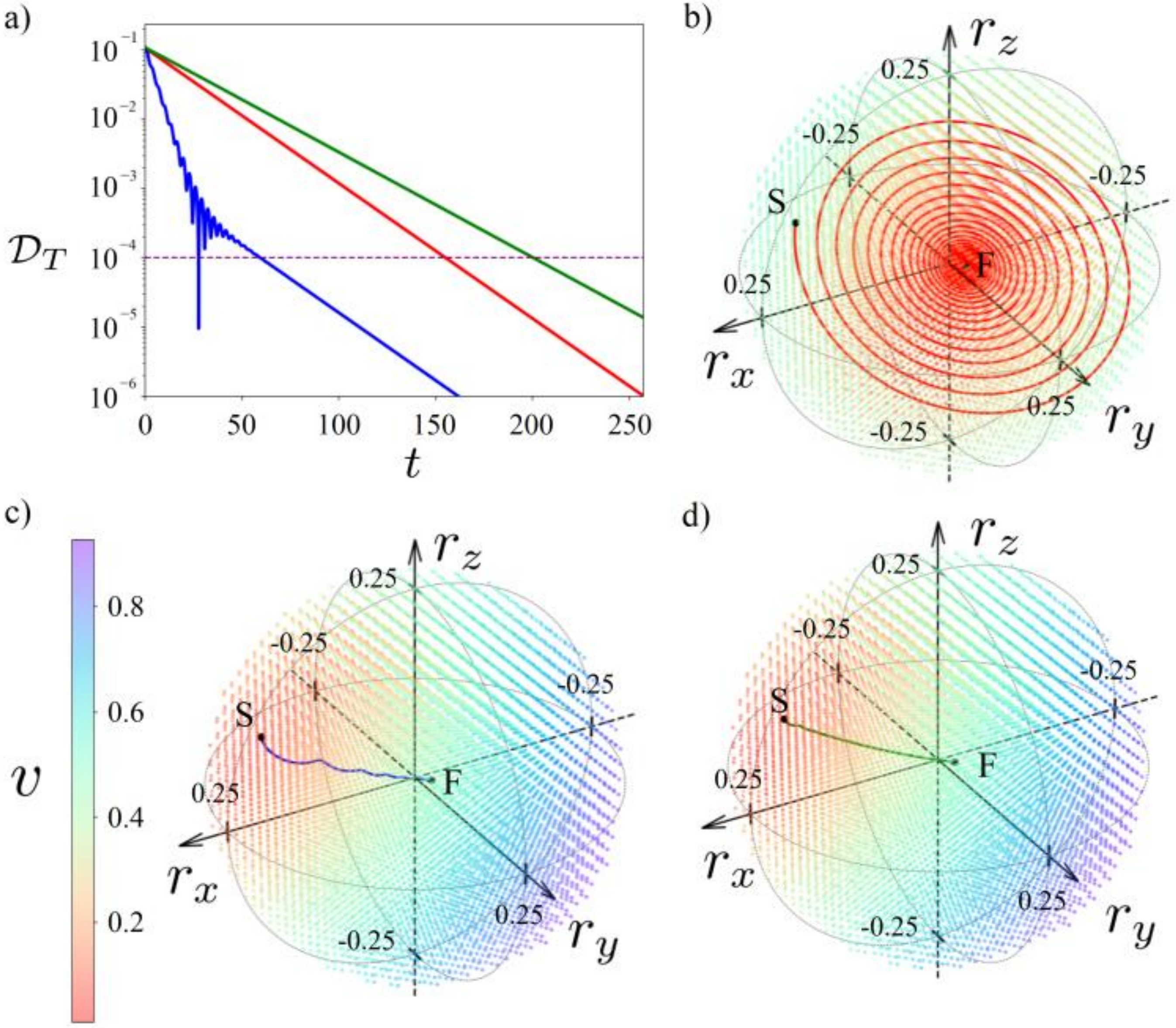
$$\gamma_\lambda(t>0) = \gamma_{\lambda,\mathrm{F}} + (\gamma_{\lambda,\mathrm{S}} - \gamma_{\lambda,\mathrm{F}})\, e^{-\kappa t} \cos(\omega t)$$

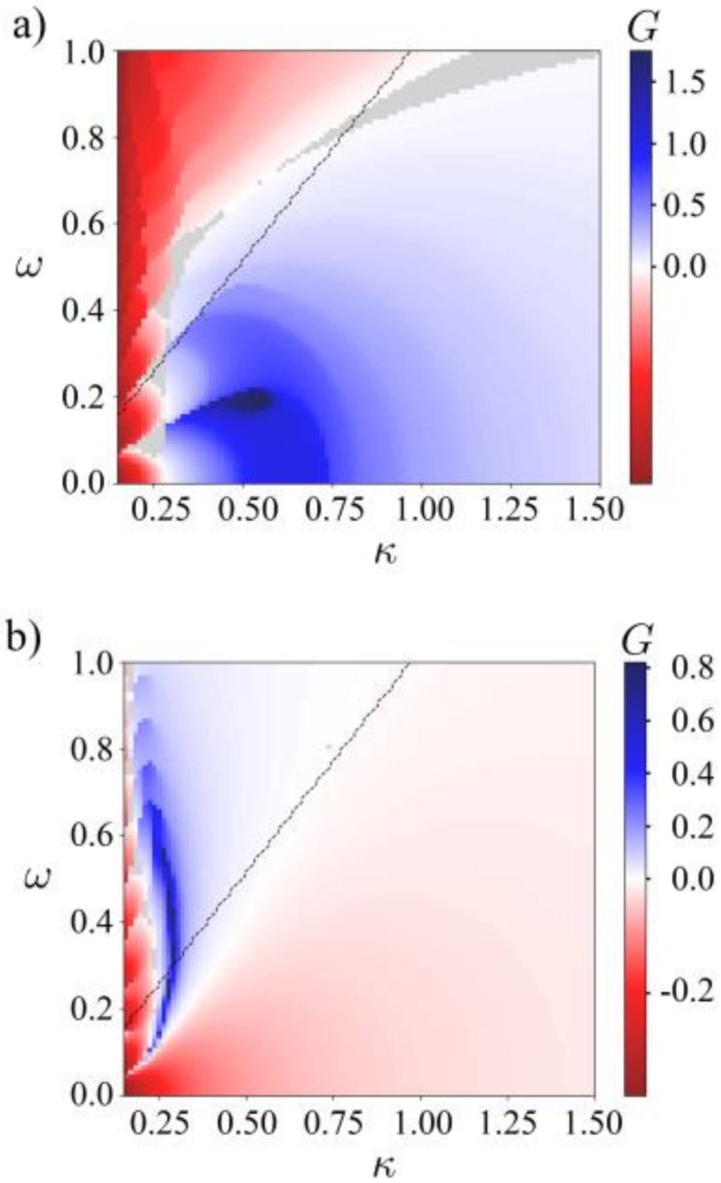
- For $\kappa \to 0$, the time evolution is quasi-static while for the sudden-quench limit $\kappa \to \infty$

- We define a "gain function"

$$G(\kappa,\omega) = \frac{\tau_{\mathrm{dir}}}{\tau_{\mathrm{cPM}}(\kappa,\omega)} - 1$$

- The gain function vanishes if both approaches (quench vs time-dependent) predict the same relaxation time, while $G < 0$ if the quench protocol is faster. If the continuous Pontus-Mpemba effect brings an advantage, we have $G > 0$


a)
b)
c)
d)
$\mathcal{D}_T$
t
0
50
100
150
200
250
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
r_z
r_x
r_y
0.25
-0.25
S
F
v
0.2
0.4
0.6
0.8


a)
b)
G
ω
κ
0.0
0.2
0.4
0.6
0.8
1.0
0.25
0.50
0.75
1.00
1.25
1.50
1.5
1.0
0.5
-0.2