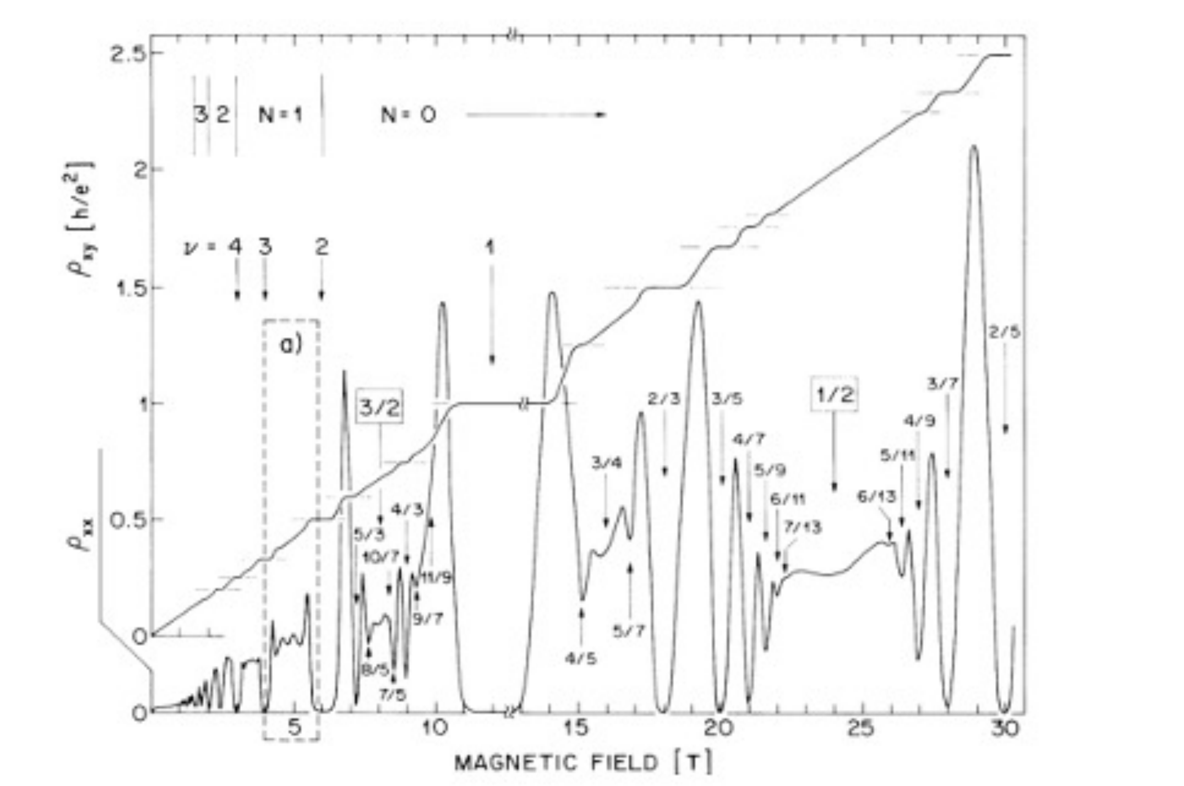
The origin

The Fractional Quantum Hall Effect

The fractional quantum Hall effect



Jain series:

Filling fraction $\nu = \frac{p}{2p+1}$

$$\rho_{xy} = \frac{2p+1}{p} = 2 + \frac{1}{p}$$

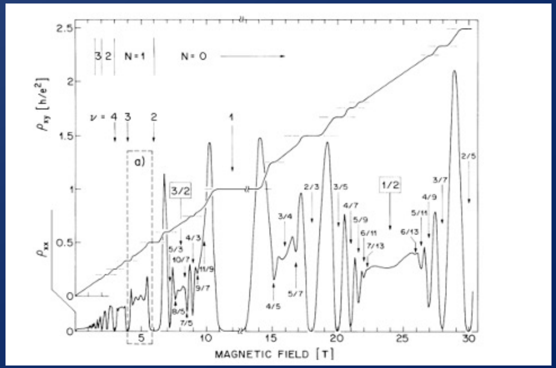
Quantized Hall resistivity ρ_{xy} and conductivity σ_{xy}

Bulk energy gap $\rho_{xx} = \sigma_{xx} = 0$

The fractional quantum Hall effect

– an endless source of scientific beauty

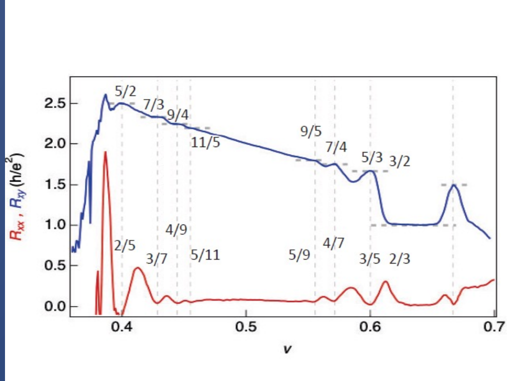
- Incredible precision
- Interaction-based, yet perturbation theory is insufficient, since the FQHE is not adiabatically connected to non-interacting electrons
- Anyons - Excitations that carry a fractional charge and follow anyonic statistics
- Bulk-edge correspondence
- …



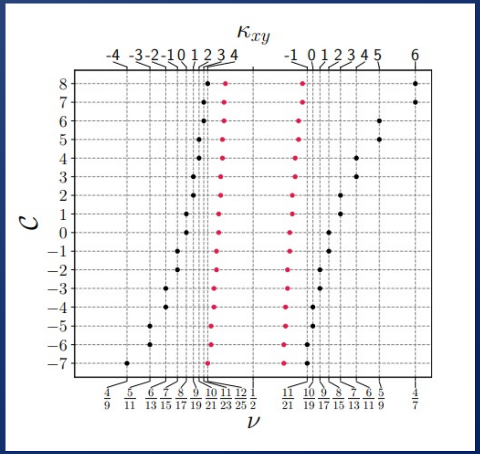
Recent developments in the study of fractionalized states of matter (a biased list)

- The Fractional Chern Insulator

(With Fu, Yue, Paul, Aboulkomsan, Knap, Pichler)



- The Fractional Excitonic Insulator
  (with Gassner, Divic, Kane)

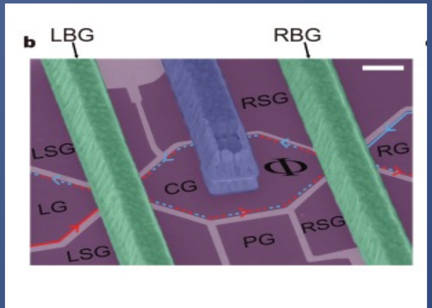


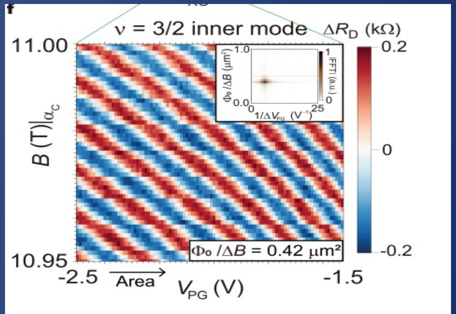
- The “Daughter States” – a new series of FQH states

(with Zheltonozhkii and Lindner)

- Observing anyons by interferometry

(with Rosenow, Shaer, Kim, Ronen…)

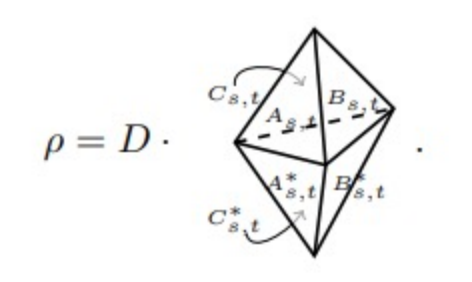




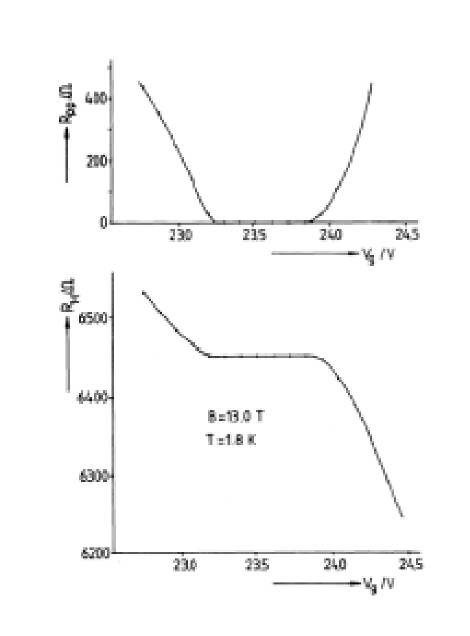
- Identifying a wave function of a fractionalized state

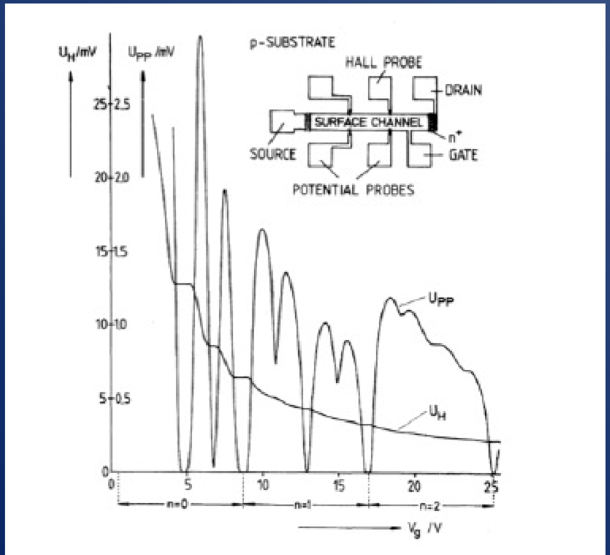
Given a wave function, what is the fractionalized state that it is the ground state of?

(with Sheffer and Berg)

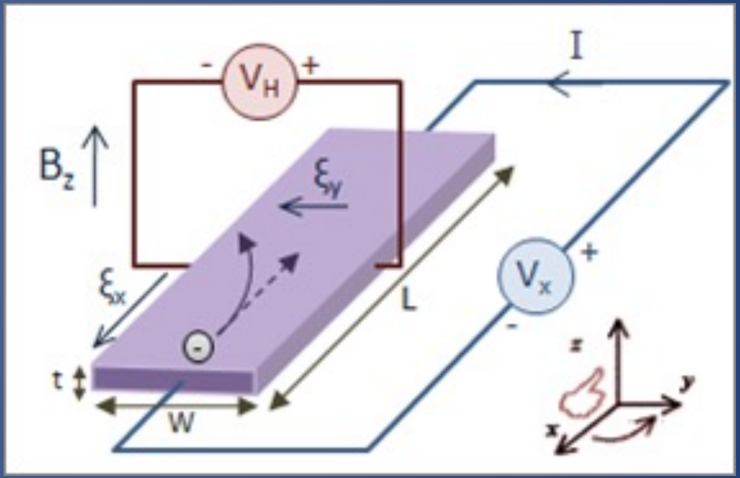


The Fractional Chern Insulator

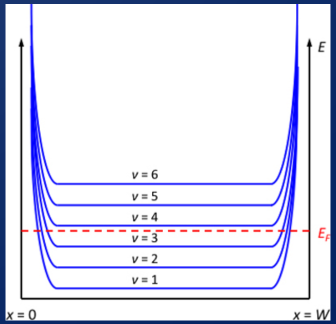


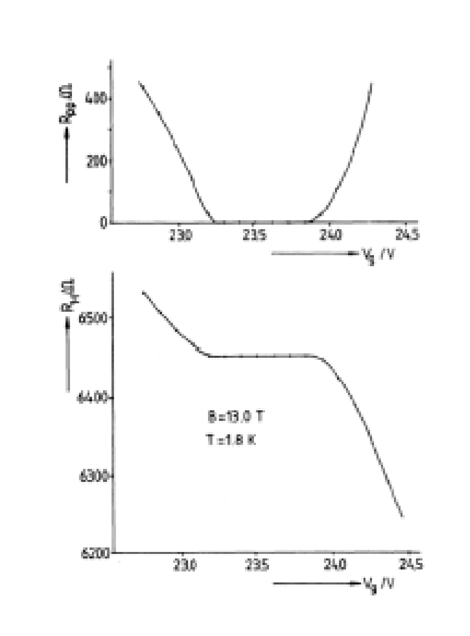


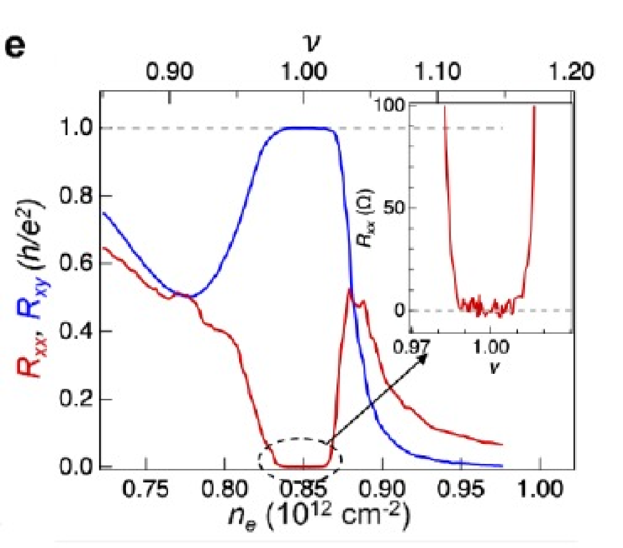
Von Klitzing, Dorda and Pepper (1980)



- Quantization of the Hall resistivity in units of h/ne^2
- Vanishing longitudinal resistivity (energy gap)
- Energy gap between Landau levels
- Magnetic field 13T







- Quantization of the Hall resistivity in units of h/ne^2
- Vanishing longitudinal resistivity (energy gap)

- ~~Energy g etw andau levels~~
- ~~Magnetic~~

Zero magnetic field

Semi-classical physics of the Hall effects

"Traditional" Hall effect

- Originating from Lorenz force $m\dot{v} = ev \times B + eE - \frac{mv}{\tau}$
- Ohm's law $E = \rho J$

With no disorder $\rho = \begin{pmatrix} \frac{im\omega}{ne^2} & \frac{B}{nec} \\ -\frac{B}{nec} & \frac{im\omega}{ne^2} \end{pmatrix}$

Resistivities are added

Excitation mode at $\omega = \omega_c$ - the cyclotron mode

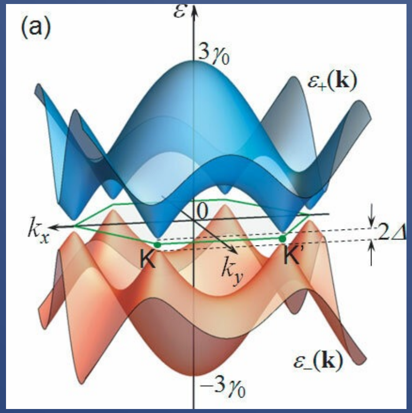
At $q \neq 0$, this is the plasma mode, at

$$\omega^2 = \omega_c^2 + \frac{n}{m} q^2 V_c(q)$$

Gapped by the magnetic field.

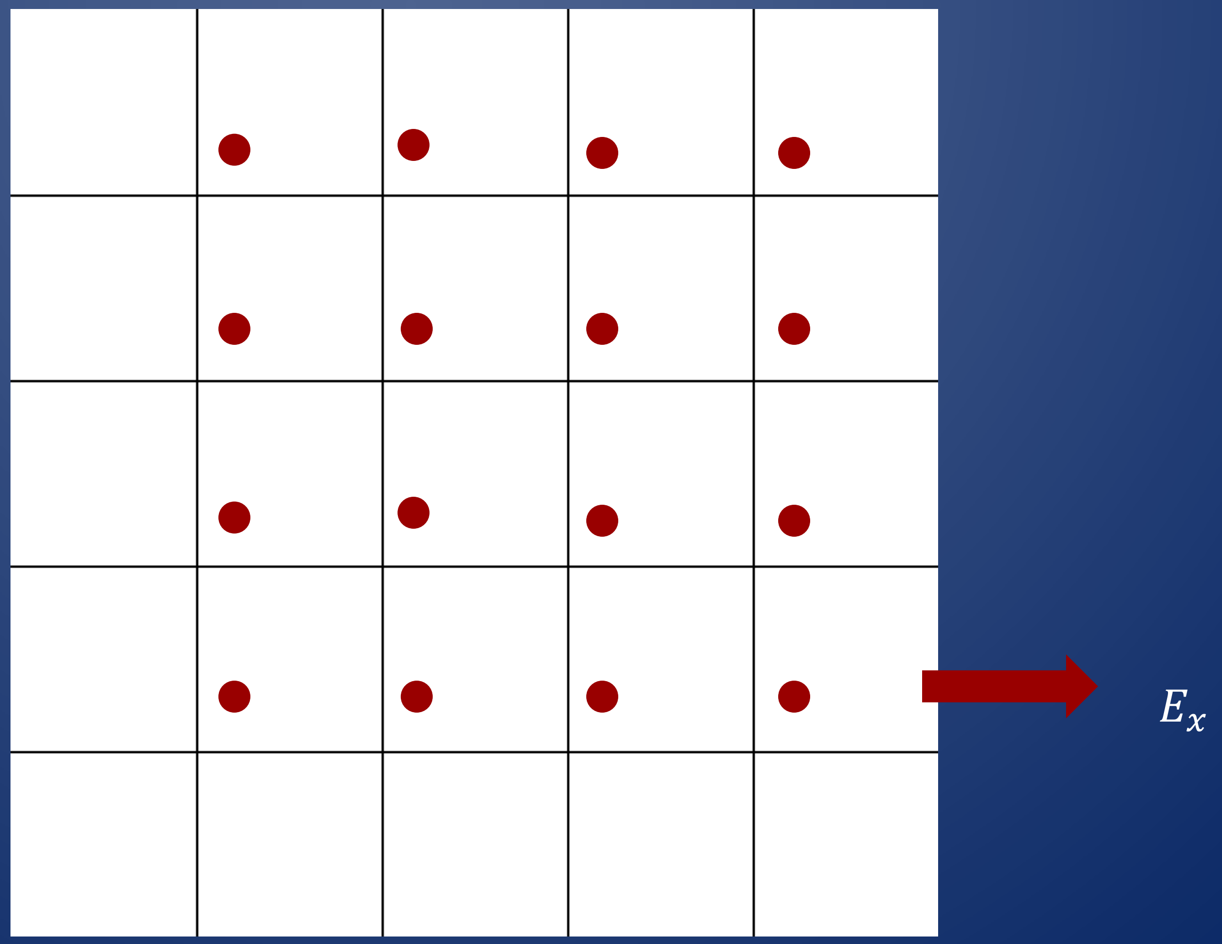
Anomalous Hall effect

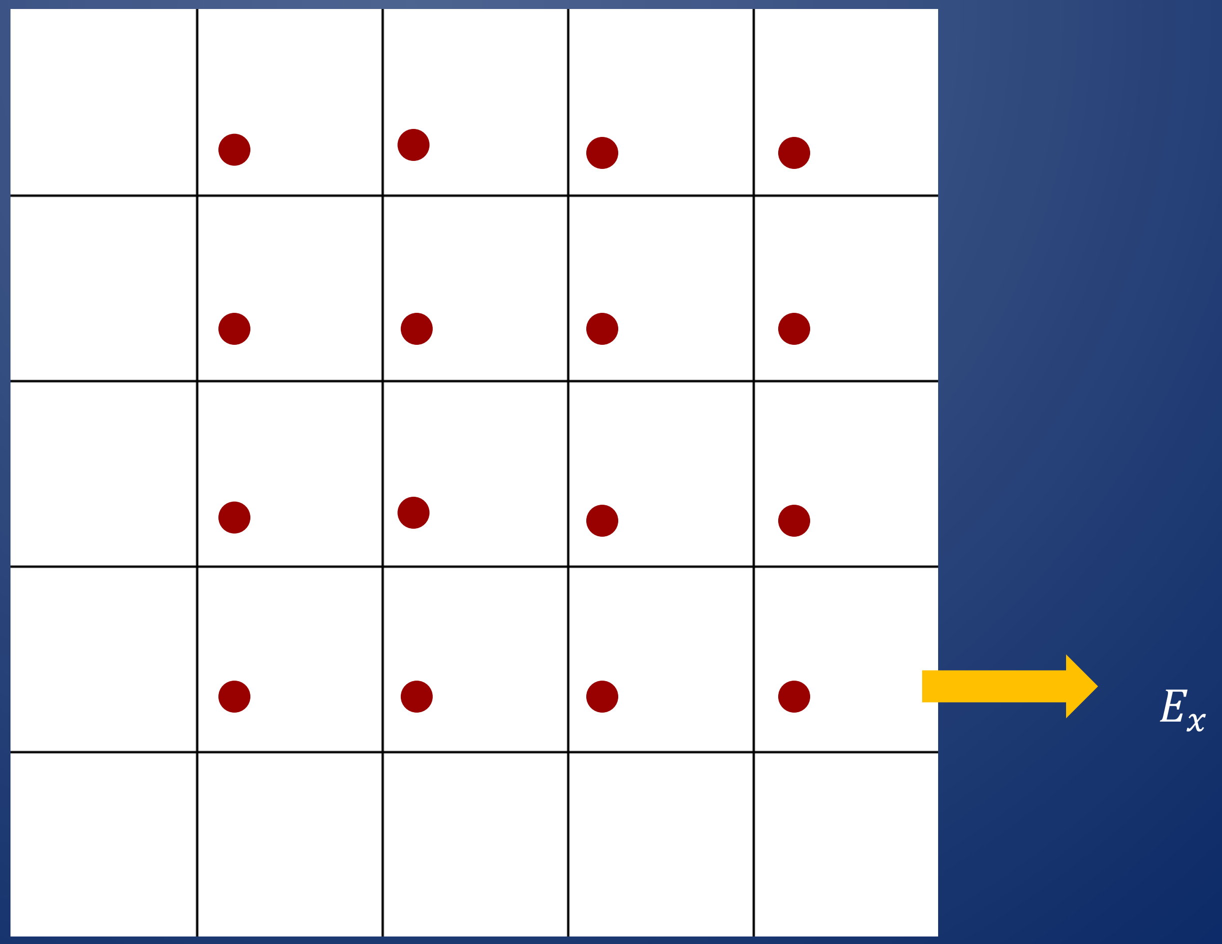
- Electrons live on a lattice
- Spectrum of energy bands: $k, \quad \epsilon(k), \quad \psi_k = e^{ik\cdot r}u_k$



- Periodicity in k – the Brillouin zone
- $u_k(r)$ describes the wave function within a unit cell of the lattice.
- Under an electric field $\hbar\dot{k} = eE.$

What happens when k comes back to itself?


E_x


E_x

Anomalous Hall effect

Semiclassical equations of motion

- $\hbar\dot{k} = eE$
- $\dot{r} = \frac{1}{\hbar}\nabla_k \epsilon + eE\times\Omega(k)$ $\qquad\qquad$ $\Omega(k) = Im\langle\frac{\partial u}{\partial k_x}|\frac{\partial u}{\partial k_y}\rangle$

Berry curvature

Anomalous Hall conductivity

Conductivity in a partially filled Chern band

Current $$e\int d^2k \left(\frac{1}{\hbar}\nabla_k \epsilon + eE\times\Omega(k)\right) f(k)$$

Add conductivities:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

$$\sigma_{xx} = -\frac{e^2}{h}\frac{\mathcal{D}}{i\omega} \qquad \sigma_{xy} = \int d^2k\Omega(k) f_0(k)$$

Anomalous Hall effect

- For a full band –

$$j_y = nev_y = \frac{1}{a_x a_y} e \frac{a_y}{T} C = \frac{1}{a_x a_y} e a_y \frac{eE_x a_x}{2\pi\hbar} C = \frac{Ce^2}{h}$$

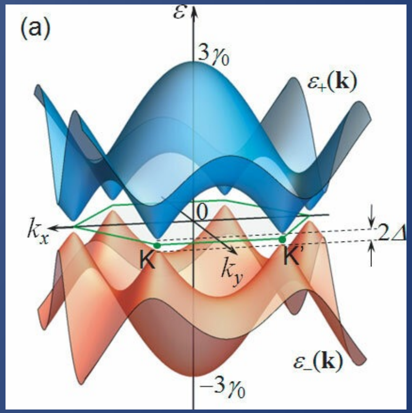
- Integer quantization of the Hall conductivity

- Every band has a Chern number

Anomalous Hall conductivity

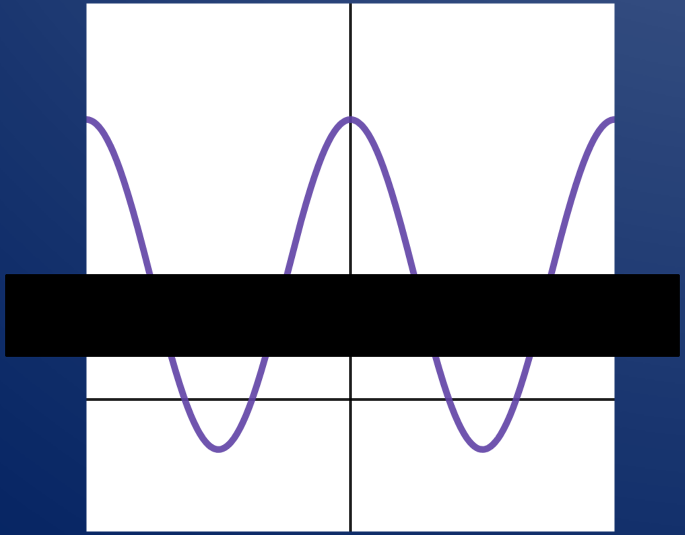
When shall we see a zero-field Integer Quantum Hall state?

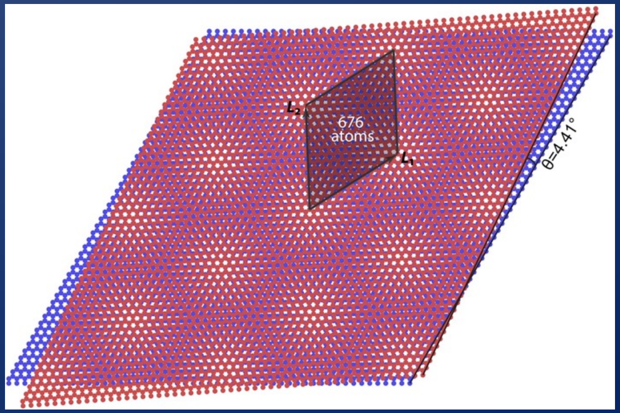
- For every band of Chern number C centered around a momentum k_0 there will be a band of Chern number $-C$ around a momentum $-k_0$



First, generate pairs of Chern bands

Introduce a larger periodicity - Moire bands

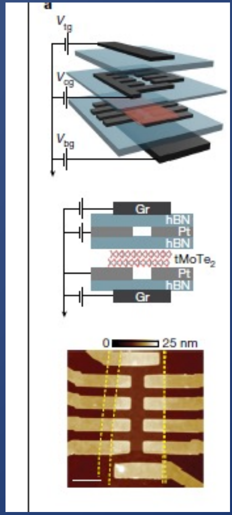


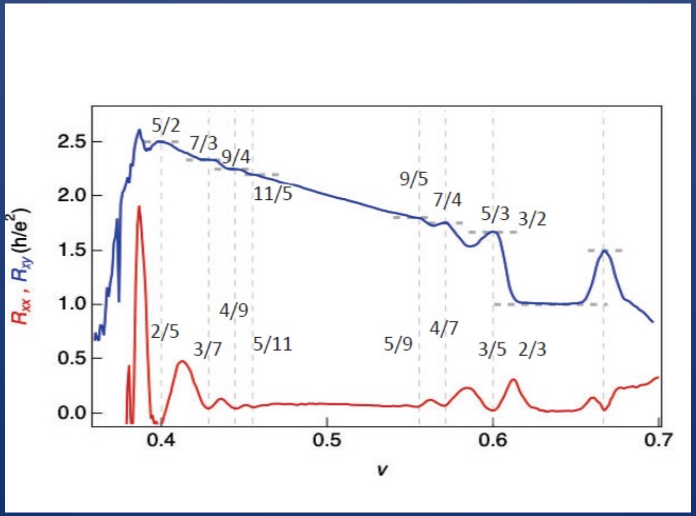


- Second, hope that the electrons will choose to spontaneously break time reversal symmetry (Hund's rule philosophy)

Recent examples:
Twisted $MoTe_2$ bi-layer
Pentalayer graphene on hBN

This is $MoTe_2$





The fractional Chern Insulator

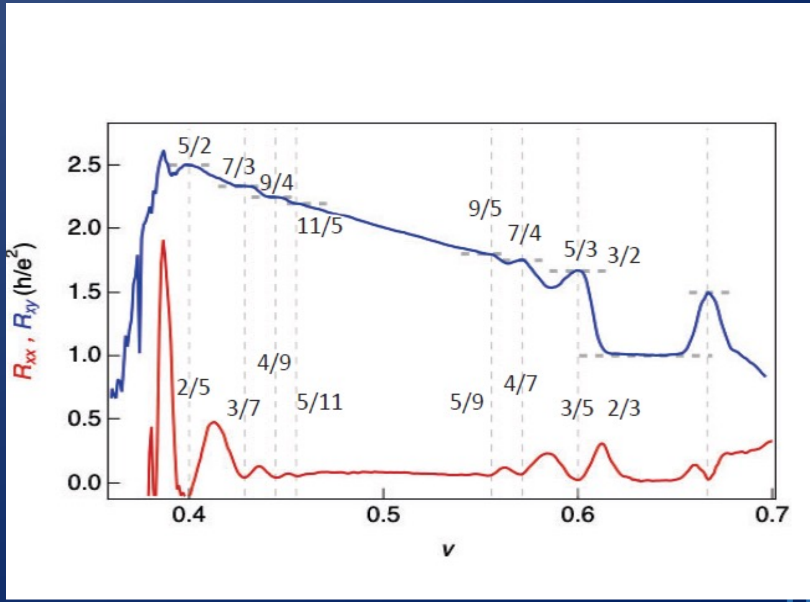
- **Fractionally quantized Hall resistivity in the absence of a magnetic field!** (Groups of Ju, Xu, Mak_Shan, Li)

- σ_{xy} is connected to the number of electrons per lattice unit cell
- Topological features carry over – interactions are essential, bulk is gapped, and edges carry gapless chiral modes.

Differences:

- Energy scales
- Absence of Galilean invariance – no Kohn's theorem

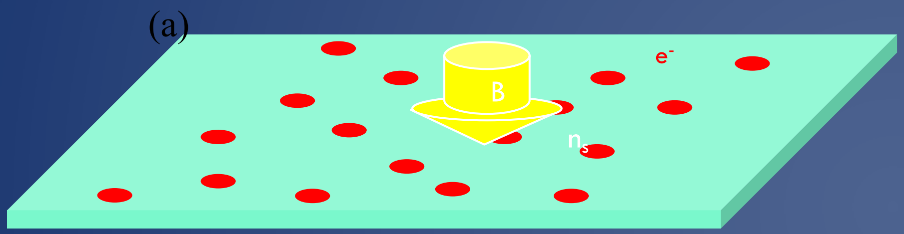
Preceding theories: Sheng's group, Neupert et al., Regnault et al. Das Sarma group

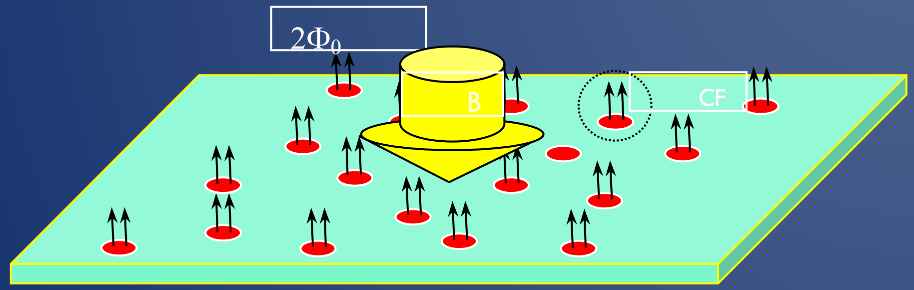


Lu et al, 2023

Flux attachment for the Fractional Quantum Hall Effect
A short review

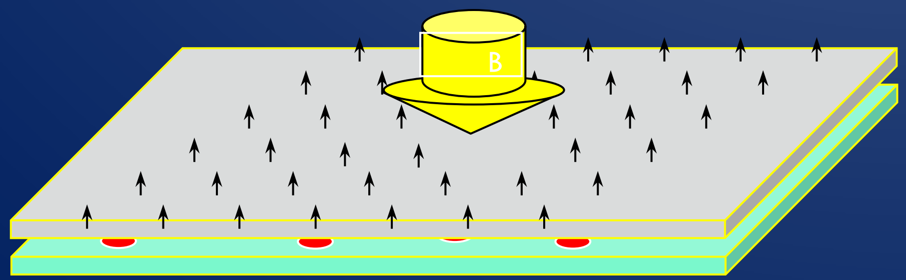
(ZHK, J, HLR, LF…)





$$\boxed{\Delta B = B - 2\phi_0 \hat{n}}$$

Mean field theory: $\Delta B = B - 2\phi_0 \langle n \rangle$



$$\rho_e = \rho_{cf} + \frac{h}{e^2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

At the mean-field level, we map interacting electrons

$$(n, B) \quad \Rightarrow \quad (n\ , \Delta B = B - 2\Phi_0 n)$$

This changes the filling factor $\nu \Rightarrow \nu_{CF}$ where $\frac{1}{\nu_{CF}} = \frac{1}{\nu} - 2$

- Fractions are mapped onto integers! $\frac{p}{2p+1} \Rightarrow p$
- Gaps appear "naturally"
- $\rho_{xy} = \frac{1}{p} + 2 = \frac{2p+1}{p}$

What do we get from this?

- A starting point for gapped states at $\nu = \frac{p}{2p+1}$
- A length scale – the cyclotron radius of the composite fermions

- An interesting choice of states at $\nu = 1/2$

$\Delta B = 0 \quad \Longrightarrow \quad$ Mapping to Composite Fermions at zero magnetic field

Flux attachment for the partially filled Chern band

Energy scales for FQH & FCI

For the fractional QHE:

Two energy scales – cyclotron and interaction, one ratio

$$\frac{e^2}{l_B}/\hbar\omega_c$$

FQHE physics already at infinitesimal interactions

For the Chern band, three energy scales – band width, band gap, electron-electron interaction

At weak interaction – Fermi liquid of electrons, no crucial dependence on the filling.

At strong interaction – FCI regime

Composite fermion theory for the fractional Chern insulator

(with Liang Fu, MIT)

- Mapping from electrons at filling factor ν at zero magnetic field to a filling factor ν at a magnetic field $2\Phi_0\nu$.
- The magnetic field is “anti-cancelled”.

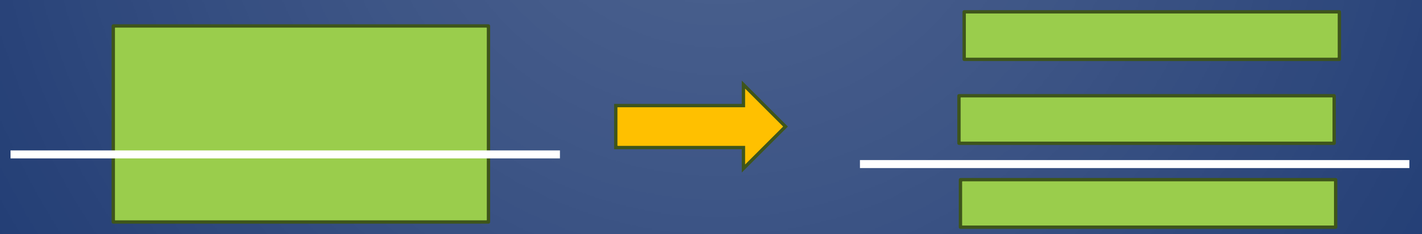
What is the effect of a magnetic field on a lattice system?

ϕ	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ

For $\phi = p/q$ the unit cell grows q-fold

$\Longrightarrow$ The band splits to q subbands, out of which p are filled

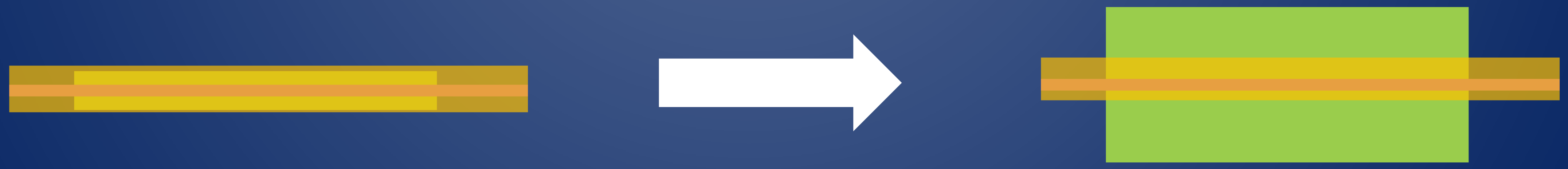
- For $\nu = p/q$ (with q odd), the unit cell grows q-fold. The band splits. A gap occurs naturally.



- For $\nu = 1/2$, the attached flux amounts to one flux quantum per unit cell $\Longrightarrow$ the band does not split.

A mapping from a half filled Chern band to a half filled (Chern-?) band.

When do composite fermions make sense for the half-filled Chern band ?



Flux attachment is there to map from a narrow band to a wide band system

$\Longrightarrow$ Phase transitions at constant filling

(Barkeshli & McGreevy, 2012)

At the half-filled Chern band:

FL Intermediate phase ? CFL

Interaction/band width

Looking at observable signatures of continuous phase transitions

(with X. Yue, M. Knap, F. Pichler)

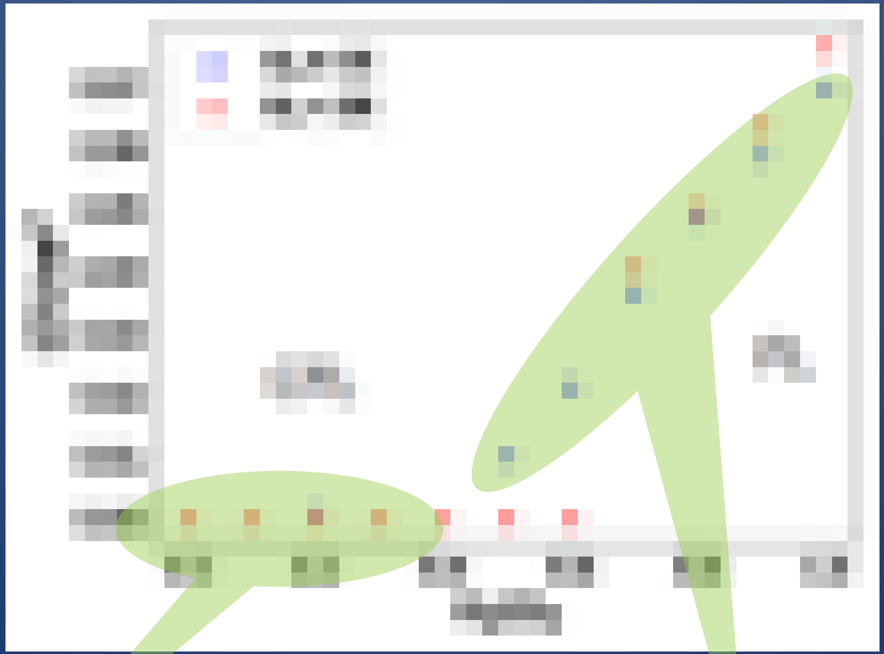
Defining properties of the CFL and the FL (clean systems!):

(AS&L. Fu, 2023)

FL: Resistivity matrix calculated within Fermi liquid theory

CFL: $\rho_e = \rho_{cf} + \frac{h}{e^2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

1. The Drude weight (the prefactor of the $1/i\omega$ term of σ_{xx}).

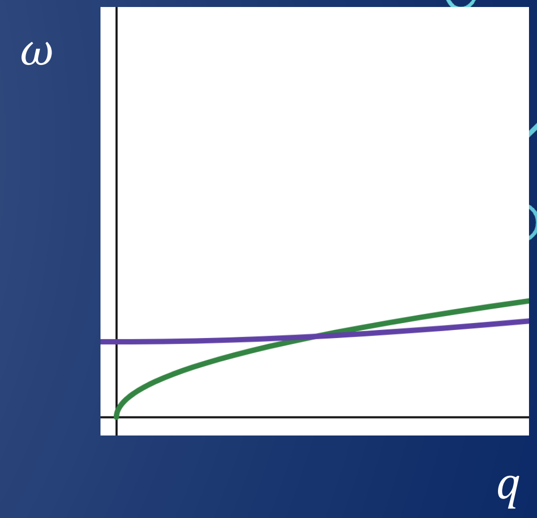


N. Paul, A. Aboulkomssan, AS, L. Fu (PRL)

$\sigma(\omega) \propto i\omega$

$\sigma(\omega) \propto \frac{1}{i\omega}$

2. Excitation modes (plasmons) at $\det \rho = 0$



Electrons in a magnetic field – cyclotron mode, gapped

Electrons in a Chern band – gapless

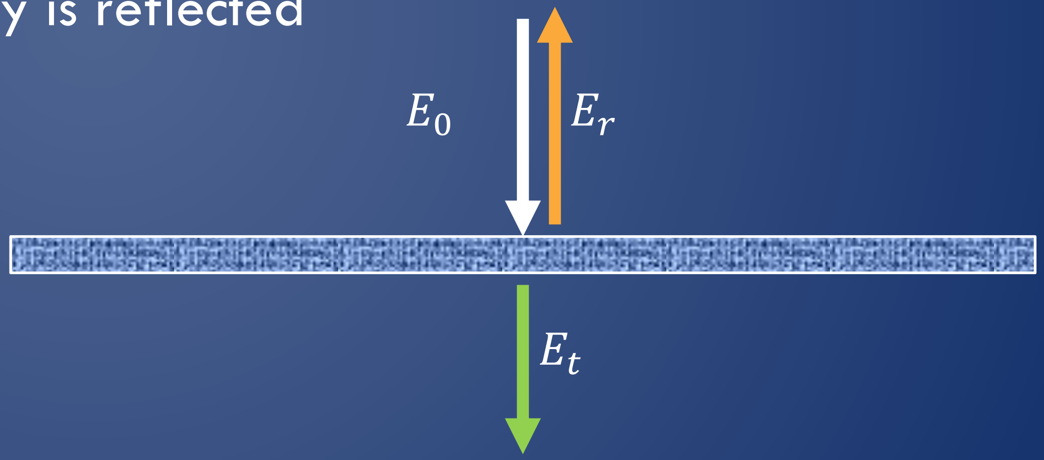
Composite fermions in a Chern band

$$\det\left[\begin{pmatrix} \frac{e^2}{h}\frac{\mathcal{D}_{CF}}{i\omega} & \sigma_{xy}^{\mathrm{CF}} \\ -\sigma_{xy}^{\mathrm{CF}} & \frac{e^2}{h}\frac{\mathcal{D}_{CF}}{i\omega} \end{pmatrix}^{-1} + \begin{pmatrix} 0 & 2h/e^2 \\ -2h/e^2 & 0 \end{pmatrix}\right] = 0$$

A gapped excitation mode at $\omega = 2\mathcal{D}^{cf}/|2\sigma_{xy}^{CF} - 1|$

Vanishing Drude weight

3. “The chiral mirror” – only one chirality is reflected



Happens when $Det\ \sigma(\omega_0) = 0$.

For A fermi liquid: $\omega_0 = \mathcal{D} \quad /|\sigma_{xy}|$

For A CFL $\omega_0 \neq 0$

$$\alpha = e^2/\hbar c$$

$$E_r = -\frac{1}{1+\alpha\sigma}\alpha\sigma E_0$$

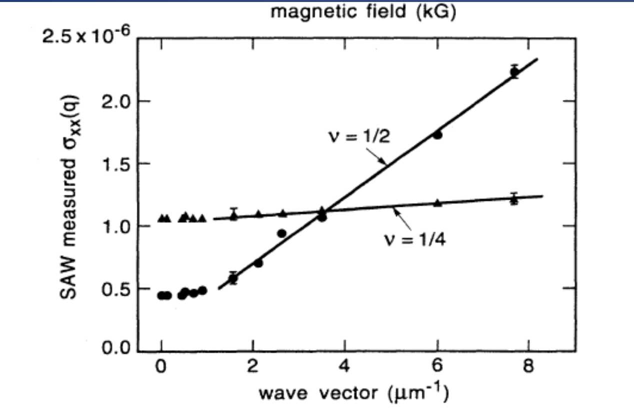
$$E_t = \frac{1}{1+\alpha\sigma}E_0$$

(Tse et al., Liu et al.)

4. The longitudinal conductivity at $q \neq 0$ (low frequency)

CFL: $\sigma_{xx} \propto q$, just as for a CFL in a strong magnetic field

FL: $\sigma_{xx} \propto \omega^2/q^3$, i.e., very small



Willett et al, 1993

The phase transitions

$$\boxed{\rho_e = \rho_{cf} + \frac{h}{e^2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}$$

ρ_{cf} - the resistivity of a band that “feels” two flux quanta

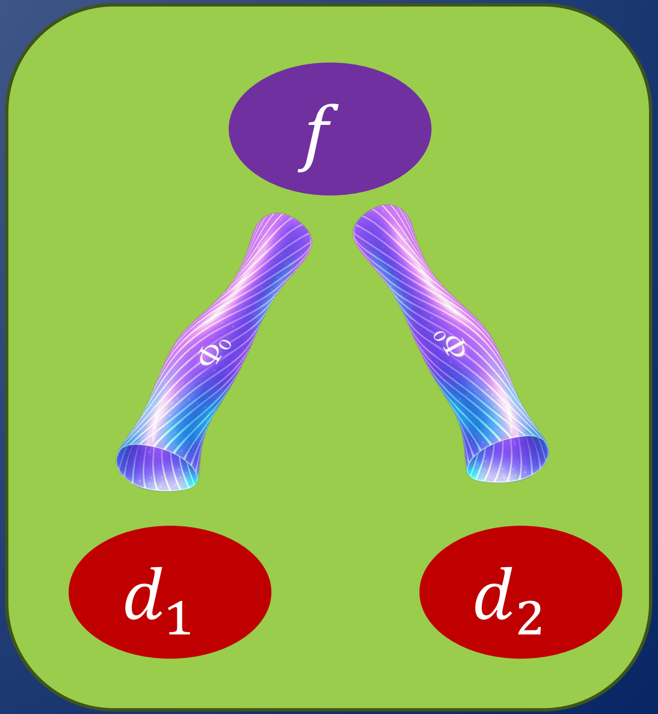
How do we turn off the flux tubes?

Parton theory:

Each electron is built of three fermionic partons: f, d_1, d_2.

f is the composite fermion, carrying the charge, feeling one flux tube from each of d_1, d_2

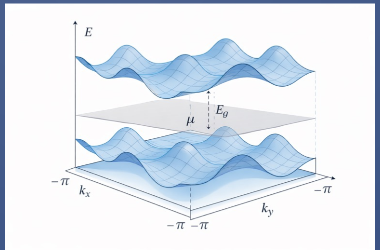
d_1, d_2 are neutral particles, each feeling one flux quantum from f.



The densities of all three are one half per unit cell.

The currents of all three are the same, leading to $\rho_e = \rho_f + \rho_{d1} + \rho_{d2}$

d_1, d_2 are gapped, f is gapless.



If d_1 and d_2 are in a $\nu = 1$ Integer Chern Insulator state, with an infinite gap to the next band, we are at the CFL phase $\rho_{d1} + \rho_{d2} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

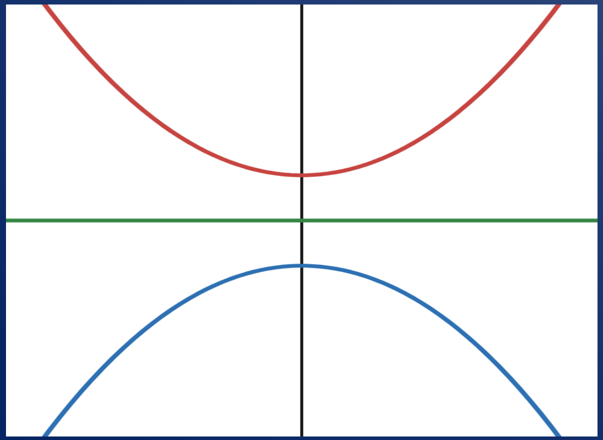
If d_1 switches to a $\nu = -1$ Integer Quantum Hall state, with an infinite gap to the next band, we are at the FL phase $\rho_{d1} + \rho_{d2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

The transition – a topological transition of the d_1 parton

For $q = 0$:

$$\rho_f = \begin{pmatrix} i\omega/\mathcal{D} & 0 \\ 0 & i\omega/\mathcal{D} \end{pmatrix}$$

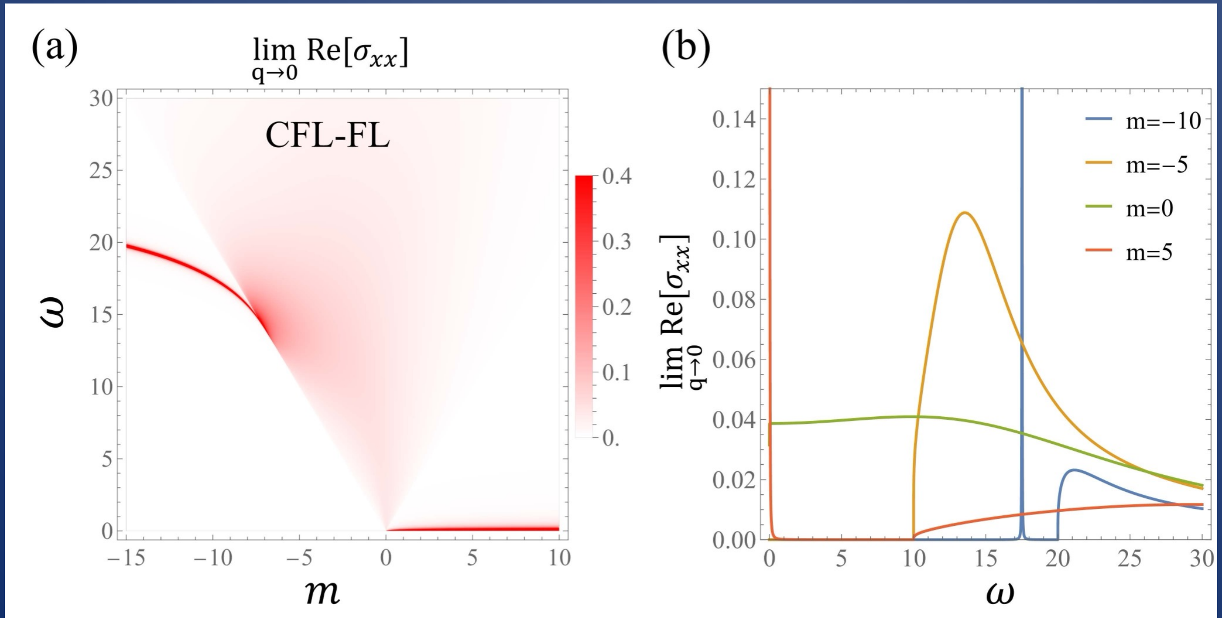
$$\rho_2 = \begin{pmatrix} 0 & h/e^2 \\ -h/e^2 & 0 \end{pmatrix}$$



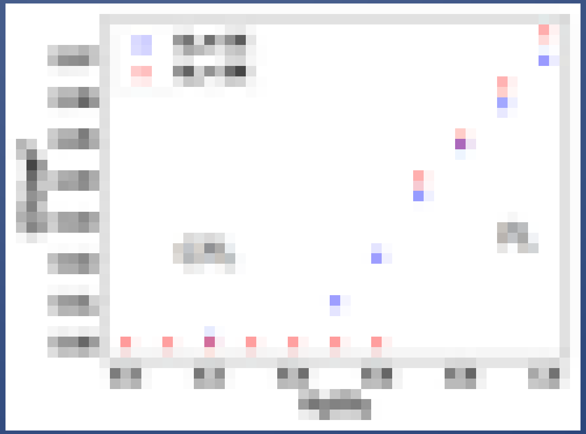
$$\sigma_1 = \begin{pmatrix} \dfrac{i\omega}{\Delta} + Re\sigma & -1 \\ 1 & \dfrac{i\omega}{\Delta} + Re\sigma \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} \dfrac{i\omega}{\Delta} + Re\ \sigma & 1 \\ -1 & \dfrac{i\omega}{\Delta} + R\ e\sigma \end{pmatrix}$$

The plasma mode is damped around the transition



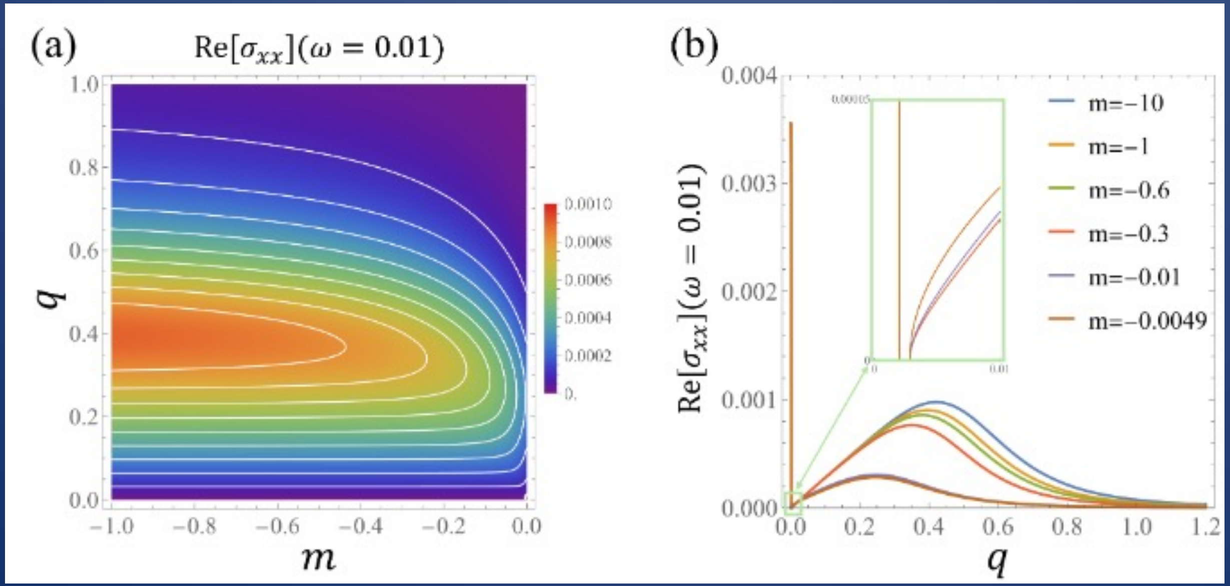
The Drude weight is zero on the CFL side, and rises linearly on the FL side



The frequency of the “chiral mirror” is continuous across the transition

$$\rho_e = \rho_f + \rho_{d1} + \rho_{d2}$$

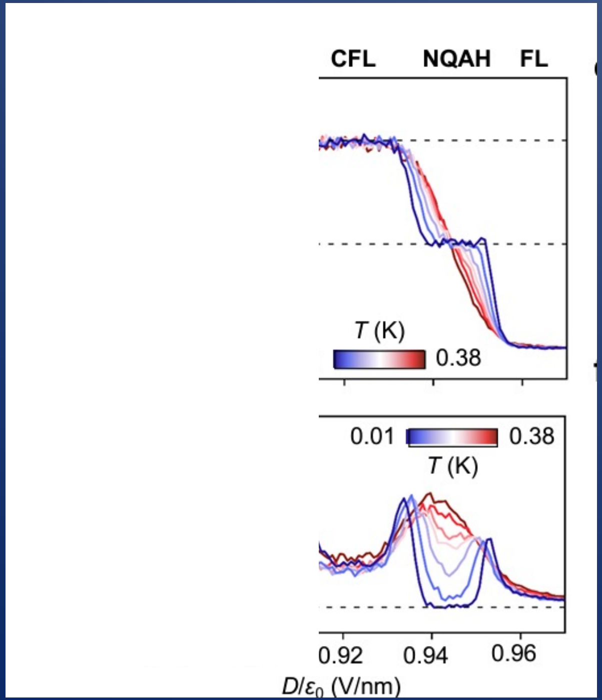
The q-linear conductivity



So far, d_1 transitioned from $C = 1$ to $C = -1$. The transition may take place through $C = 0$.

Then, d_1 is a trivial insulator, and ρ_{d1} dominates the sum over resistivities.

If the insulator is formed by the holes in the Chern band, the resulting state is a $\nu = 1$ integer Chern insulator.



Lu et al., 2024

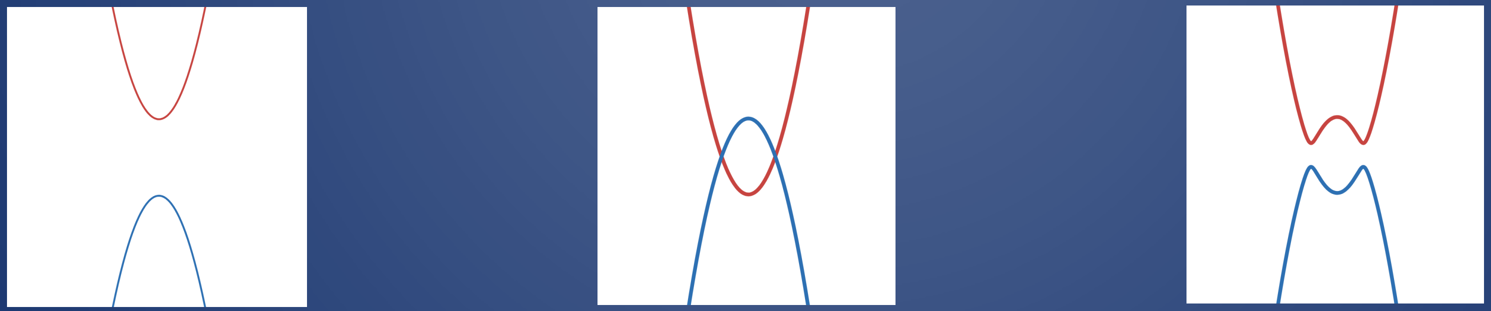
The fractional excitonic insulator – a fractional σ_{xy} at an integer filling

So far, magic fractional fillings ν led to $\sigma_{xy} = \frac{e^2}{h}\nu$.

Could the two be separated?

- Yes, with breaking of translational symmetry (e.g., Hall crystals)
- Yes, without breaking of translational symmetry (Fractional Excitonic Insulator)
  - (Venderbos, Hu, Kane 2018)

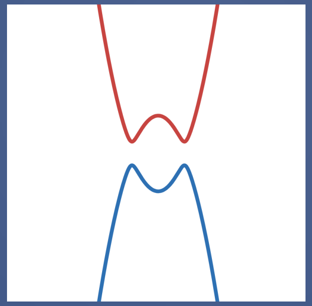
- Start from two trivial bands ($C = 0$), one full and one empty
- Band-invert and couple the two bands



- If the coupling $\left(k_x + ik_y\right)^m$ a Berry curvature $\propto m$ develops in a small part of the Brillouin zone

- In terms of the original bands - Electron-hole pairs

- Integer Quantized Hall effect (full band)



The coupling term in real space:

$$\mathrm{t}\int dk\, c^{+}(k)\left(k_x + ik_y\right)^m c(k) \;\Rightarrow \int d\delta z\, \mathrm{t}(\delta z)\psi^{+}(r+\delta z)\psi(r)$$

With $t(\delta z) \;\propto\; (\delta z)^m e^{-\frac{\delta z^2}{d^2}} \;=\; |\delta z|^m \quad e^{-|\delta z|^2/d^2} \quad e^{im\theta}$

- Attach $2\Phi_0$ to every electron and $-2\Phi_0$ to every hole

  $\Longrightarrow$ an electron-hole pair carries no flux, if they are created at the same point.

But -

$$\int d\delta z\, \mathrm{t}(\delta z)\psi^{+}(r+\delta z)\psi(r)$$

with $t(\delta z) \propto |\delta z|^m e^{-|\delta z|^2/d^2} \quad e^{im\theta}$

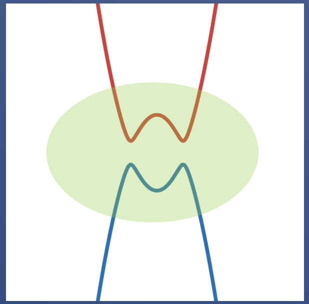
Flux attachment changes m to $m-2$, so C changes to $C-2$.

But -

$$\boxed{\rho_e = \rho_{cf} + \frac{h}{e^2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}$$

➡ Fractional state at integer filling!

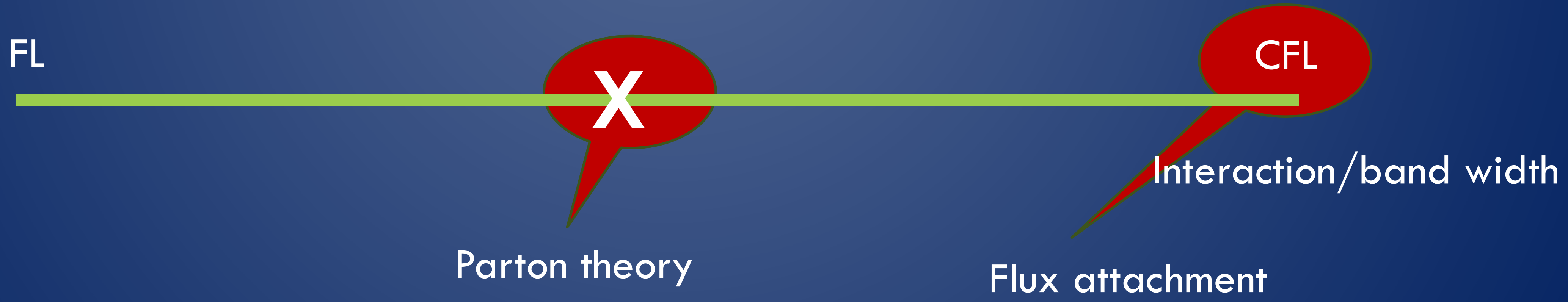
For $m = 3$, flux attachment maps to $m = 1$, leading to $\sigma_{xy} = \frac{\mathrm{e}^2}{\mathrm{h}}\frac{1}{3}$.



Electron-hole pairs in a small part of the Brillouin zone create a fractional state at a density that corresponds to a full band.

Summary:

1.



2. Fractional Excitonic insulators – fractional state at integer filling