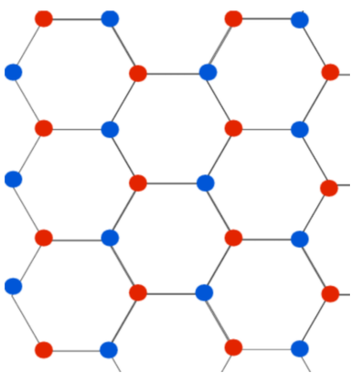
# Exotic Forms of Superconductivity

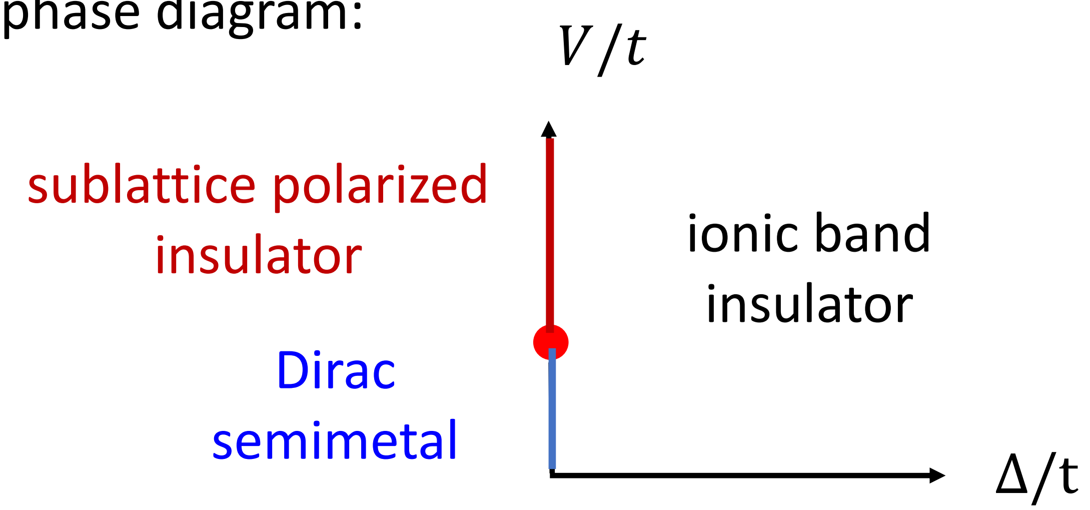
Liang Fu (MIT)

# Honeycomb Model



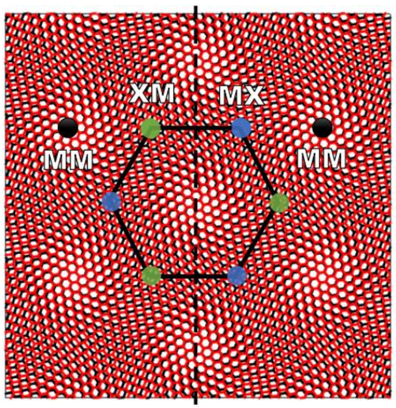
$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_t,$$
$$\mathcal{H}_0 = V \sum_{\langle r,r' \rangle} n_r n_{r'} + \Delta \sum_{r \in B} n_r,$$
$$\mathcal{H}_t = -t \sum_{\langle r,r' \rangle} (c_r^\dagger c_{r'} + hc).$$

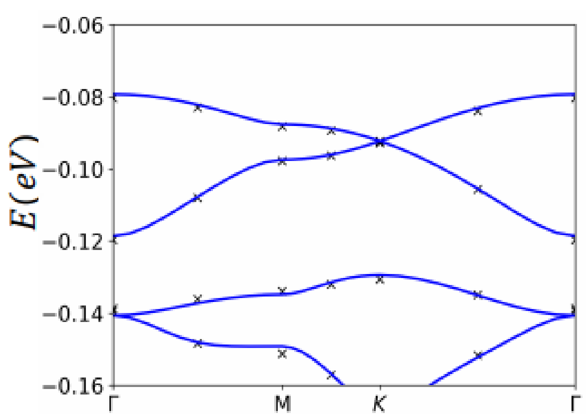
Half-filling phase diagram:

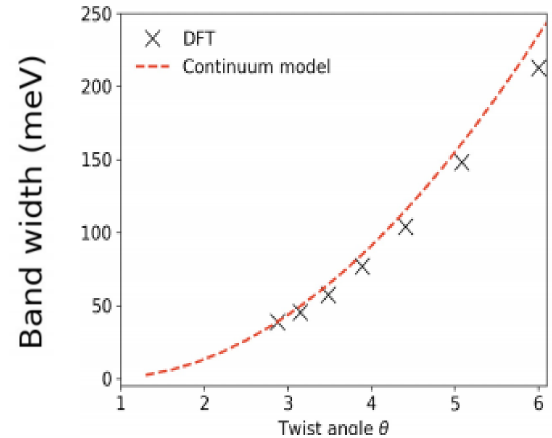


# Semiconductor Moire Materials

Twisted homobilayer $\Gamma$-valley transition metal dichalcogenide




Zhang, Liu & LF, PRB (2021)

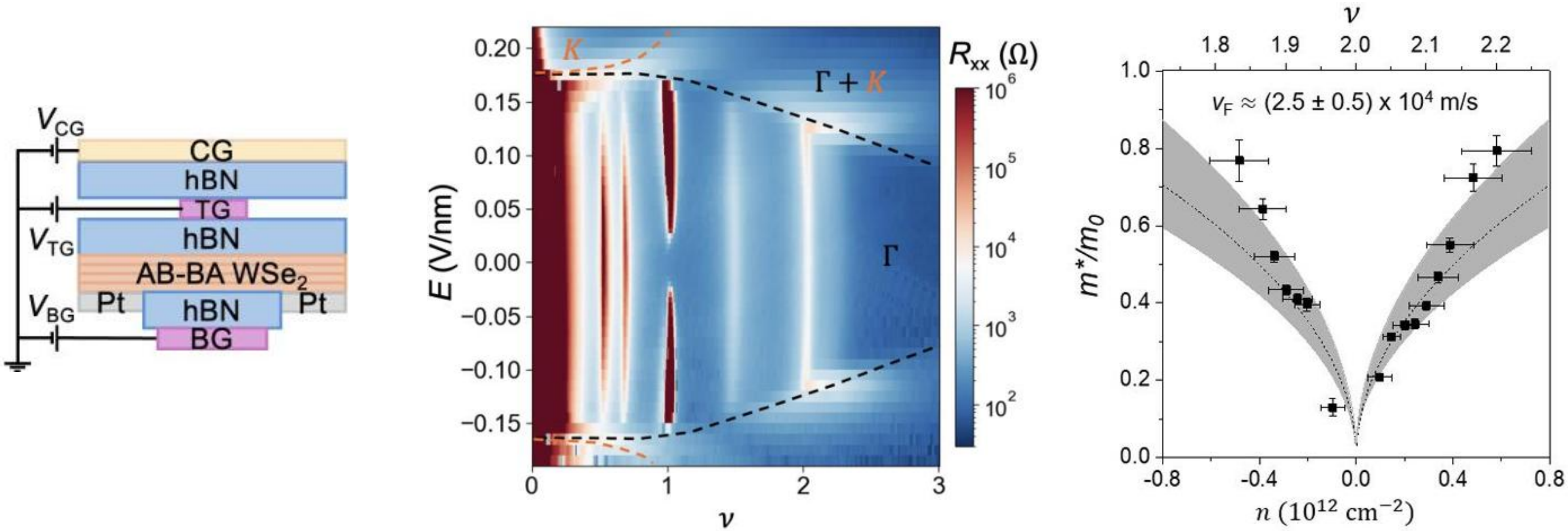
- 2D hole gas in periodic potential

$$V(\boldsymbol{r}) = -V_0 \sum_i \cos(\boldsymbol{G_i} \cdot \boldsymbol{r} + \phi)\,, \quad G \sim a/\theta$$

- potential minima at XM and MX sites forming honeycomb lattice
- narrow band at small twist angle $\hbar^2 G^2/m \ll V_0$

# Relativistic Mott transition in strongly correlated artificial graphene

Liguo Ma[1,2], Raghav Chaturvedi[1], Phuong X. Nguyen[1,3], Kenji Watanabe[4], Takashi Taniguchi[4], Kin Fai Mak[1,3,5*], Jie Shan[1,3,5*]




- Electric field introduces potential bias between XM and MX sites, realizing tunable honeycomb lattice.
- In-plane B field induces spin polarization near $\nu = 1$.

# SC in Doped Ionic Insulator: $\Delta \gg t$


Crepel & LF, Sci Adv (2021)

# Effective Hamiltonian for Doped Particles

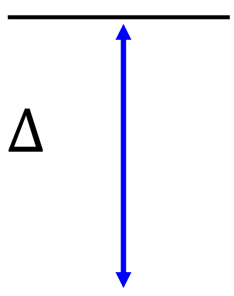
For $\Delta \gg t$, low-energy subspace: $N_A = N, N_B = N_p$

high-energy subspace: $N_A < N, N_B = N_p + (N - N_A)$

with $N_p$ the number of doped particles.

Kinetic hopping between nearest neighbor sites changes $N_A - N_B$, thus coupling low- and high-energy subspaces.

$E_g \geq \Delta$

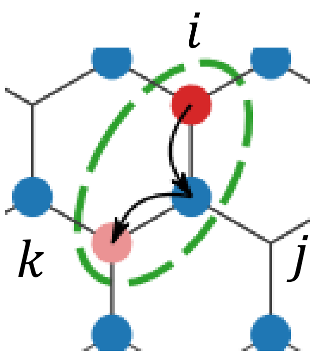


$$\mathcal{H}_0 = V \sum_{\langle r,r' \rangle} n_r n_{r'} + \Delta \sum_{r \in B} n_r$$

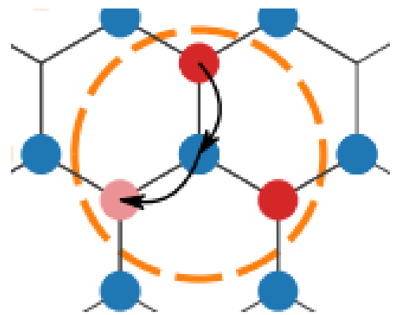
$$\equiv H_l \bigoplus H_h$$

Crepel & LF, Sci Adv (2021)

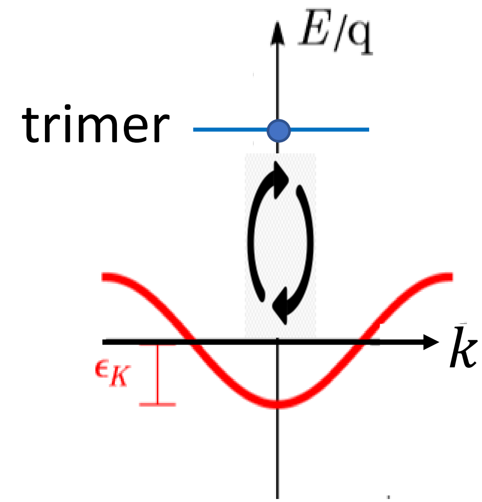
# Trimer Mediated Attraction




$E_g = \Delta + V$



$E_g = \Delta$




Transition amplitude of a doped particle from site $i$ to $k$ depends on the occupation of nearby site $j$.

single-particle hopping: $t_f(f_i^\dagger f_j + hc)$

$$t_f = \frac{t^2}{\Delta + V},$$

correlated hopping: $\lambda(f_i^\dagger n_j f_k + P_{ijk})$

$$\lambda = \frac{t^2}{\Delta} - \frac{t^2}{\Delta + V}$$

# Effective Hamiltonian for Doped Particles

Decouple low- & high-energy manifold by Schrieffer-Wolf transformation

$$\mathcal{H}' = e^{iS}\mathcal{H}e^{-iS} \qquad H_l' = H_l + \sum_n \left(\frac{t}{\Delta}\right)^n \delta H_n$$

After performing SW, off-diagonal part is removed and diagonal part acquires corrections that can be calculated perturbatively.

$$\mathcal{H}' = \sum_{\langle i,j \rangle} t_f (f_i^\dagger f_j + hc) + V_f n_i n_j + \sum_{(ijk)\in\triangle} \lambda (f_i^\dagger n_j f_k + P_{ijk}) + U_3 n_i n_j n_k .$$

$$t_f = \frac{t^2}{\Delta + V}, \qquad V_f = -\frac{t^2}{\Delta} + \frac{4t^2}{\Delta + V} - \frac{3t^2}{\Delta + 2V},$$

$$\lambda = \frac{t^2}{\Delta} - \frac{t^2}{\Delta + V} \qquad U_3 = \frac{3t^2}{\Delta} - \frac{6t^2}{\Delta + V} + \frac{3t^2}{\Delta + 2V}.$$

Crepel & LF, Sci Adv (2021)

# Effective Hamiltonian for Doped Particles

Decouple low- & high-energy manifold by Schrieffer-Wolf transformation

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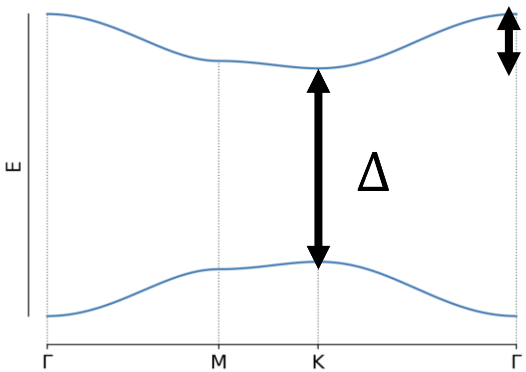
$$\mathcal{H}' = \sum_{\langle i,j \rangle} t_f (f_i^\dagger f_j + hc) + V_f n_i n_j + \sum_{(ijk)\in\triangle} \lambda (f_i^\dagger n_j f_k + P_{ijk}) + U_3 n_i n_j n_k .$$

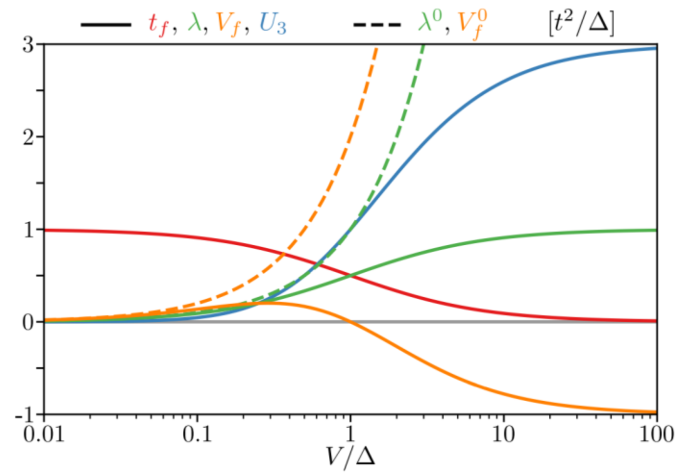
- exact at large $\Delta/t$ for any doping & interaction
- effective interaction is instantaneous and finite-ranged.

# Interband Screening

Noninteracting bands




$W = 9t^2/\Delta$



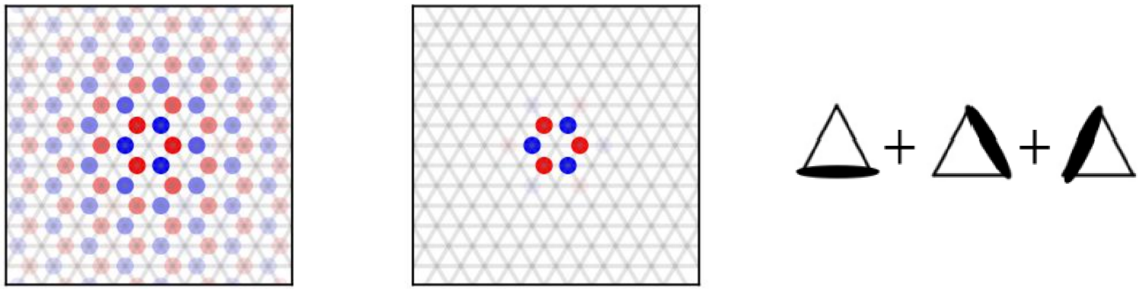

- conduction band minima at $\pm K$: valley degeneracy
- effective interaction is *strongly* modified by interband processes
- interaction projected to partially filled band misses the key: interband screening responsible for attraction!

# Two-Particle Bound State

## Attraction from repulsion

f-wave pairing

a)




$$\mathcal{H}' = \sum_{\langle i,j\rangle} t_f (f_i^\dagger f_j + hc) + V_f n_i n_j + \sum_{(ijk)\in\triangle} \lambda (f_i^\dagger n_j f_k + P_{ijk})$$

- For large $V/\Delta$, only virtual transition to trimer occurs, leading to tight bound pair resonating within a triangle (f-wave).

$$t_f = 0,\ \lambda = -V_f = t^2/\Delta, \quad \epsilon_b = 3t^2/\Delta$$

- For small $V/\Delta$, pair size is large.

# Continuum Theory at Low Density

$$\mathcal{H}' = \sum_{\langle i,j \rangle} t_f (f_i^\dagger f_j + hc) + V_f n_i n_j \; + \sum_{(ijk)\in\triangle} \lambda(f_i^\dagger n_j f_k + P_{ijk}) + U_3 n_i n_j n_k.$$

Focusing on low-energy modes around $\pm K$

$$f_r = \frac{1}{\sqrt{N_s}} \sum_{\tau=\pm} \sum_{k,\, ka \ll 1} e^{-i[(\tau K + k)\cdot r]} \psi_{\tau,k}$$ (coarse graining)

$$\widetilde{\mathcal{H}} = \int \mathrm{d}x \sum_{\tau=\pm} \psi_\tau^\dagger \left[\frac{-\nabla^2}{2m}\right] \psi_\tau + g\psi_+^\dagger \psi_+ \psi_-^\dagger \psi_-,$$

$$m = 2/(3t_f a^2),$$
$$g = 6a^2(V_f - 2\lambda) < 0.$$

Two-component Fermi liquid with local attraction:

- BCS-BEC crossover tuned by interaction and density
- s-wave valley-singlet = f-wave pairing on the lattice

# Two-Particle Bound State

$$\widetilde{\mathcal{H}} = \int \mathrm{d}x \sum_{\tau=\pm} \psi_\tau^\dagger \left[\frac{-\nabla^2}{2m}\right] \psi_\tau + g\psi_+^\dagger \psi_+ \psi_-^\dagger \psi_-,$$

Binding energy: $$\frac{1}{g} = -\int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{\varepsilon_b + |q|^2/m}$$

$$\varepsilon_b = \frac{\Lambda^2}{m\left[e^{\frac{4\pi}{m|g|}} - 1\right]} \Rightarrow \epsilon_b \propto \frac{9W}{\pi}\left(e^{1/g} - 1\right)^{-1}$$

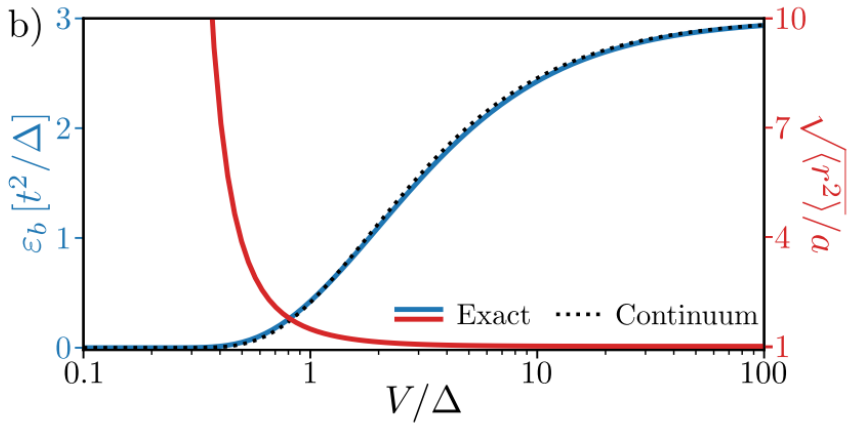
with $W = 9t_f = \frac{9t^2}{\Delta+V}$, $g = \frac{6V^2}{\pi\Delta(\Delta+2V)}$

- UV cutoff $\Lambda = \sqrt{2\pi/3a^2}$ ($\Lambda^2/m = \pi t_f$) chosen so that $\epsilon_b$ agrees with lattice calculation at $V \gg \Delta$.

# Two-Particle Bound State

from **exact** solution of effective model $H'$

Pair binding energy $\epsilon_b$




Pair size

$$\epsilon_b \propto \frac{9W}{\pi}\left(e^{1/g_0} - 1\right)^{-1} \quad \text{with } W = \frac{9t^2}{\Delta+V},\ g_0 = \frac{6V^2}{\pi\Delta(\Delta+2V)}$$

(nonperturbative in $V$)

New dimensionless coupling emerges: $g_0$ only depends on $V/\Delta$.
$g_0$ can be small even when $V \gg t$.

# “Weak”-Coupling, yet Non-BCS

For small $g_0$, attractive Fermi gas is superconducting.

$$1 = -\sum_k V \frac{\tanh(\epsilon_k / 2T_c)}{2\epsilon_k}$$

Express interaction strength in terms of two-particle binding energy

$$\sum_k \frac{1}{\epsilon_k + \epsilon_b} = \sum_k \frac{\tanh(\epsilon_k / 2T_c)}{2\epsilon_k}$$

=> $k_B T_c = e^{\gamma - 1} \sqrt{2 E_F \varepsilon_b} / \pi$ Miyake (1983)

No Debye cutoff

# “Weak”-Coupling, yet Non-BCS

For small $g_0$, our model at low doping is superconducting with

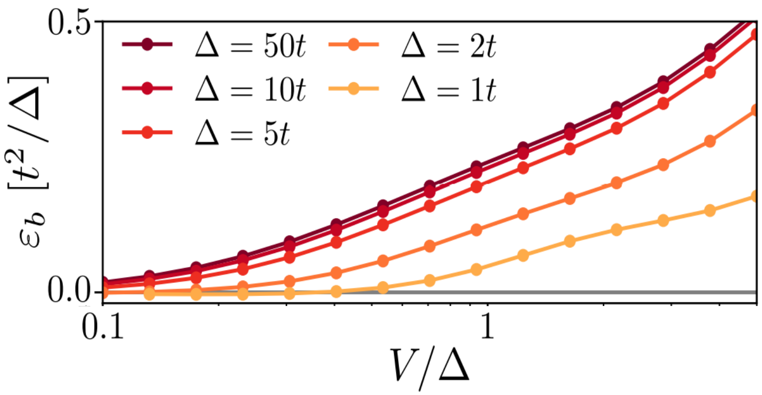
$$T_c, \Delta \propto \sqrt{\delta}\, W e^{-\frac{1}{2g_0}}$$

- accurate for $T_c \leq 0.1 E_F$ (known from BCS-BEC crossover)
- $\frac{\Delta}{k_B T_c} \approx \mathbf{4.79}$ (1.764 in phonon SC with Debye cutoff)
- $T_c \propto \sqrt{\delta}$ despite constant density of states

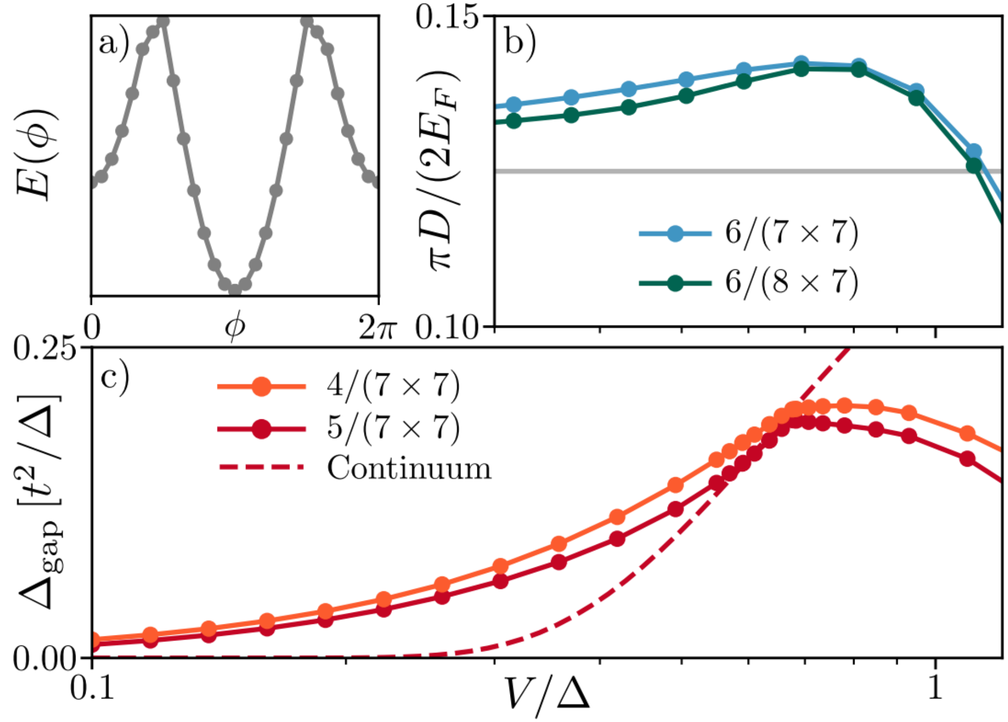
BCS-BEC crossover at low density ($E_F < \epsilon_b$, strong-coupling regime)

# Exact Diagonalization

## Two-particle binding energy




## SC at finite doping density



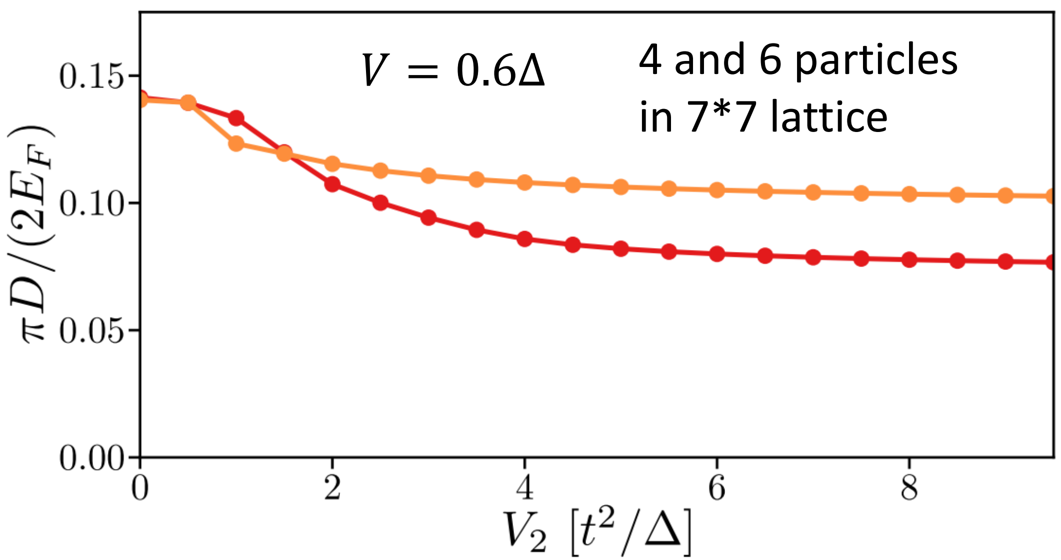

$$\Delta_{\mathrm{gap}} = \frac{(-1)^N}{2}\left[E(N+1) + E(N-1) - 2E(N)\right]$$

$$D = \frac{1}{16\pi^2}\,\frac{L_1}{L_2}\,\frac{\partial^2 E(N,\phi)}{\partial \phi^2}\bigg|_{\phi=0}$$

# Longer-Range Interaction

$$\mathcal{H}' = \sum_{\langle i,j \rangle} t_f (f_i^\dagger f_j + hc) + V_f n_i n_j \; + \sum_{(ijk)\in\triangle} \lambda (f_i^\dagger n_j f_k + P_{ijk}) + U_3 n_i n_j n_k .$$

For $V_2 \ll \Delta$, $\quad V_f' = V_2 + V_f$ .

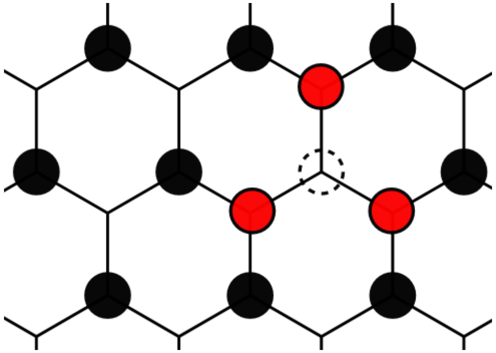


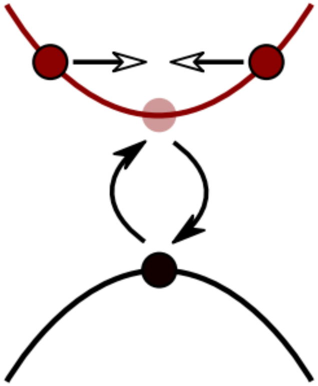

SC at finite doping is robust against long-range interaction.

# Summary

Excitonic mechanism for SC from repulsive interaction:

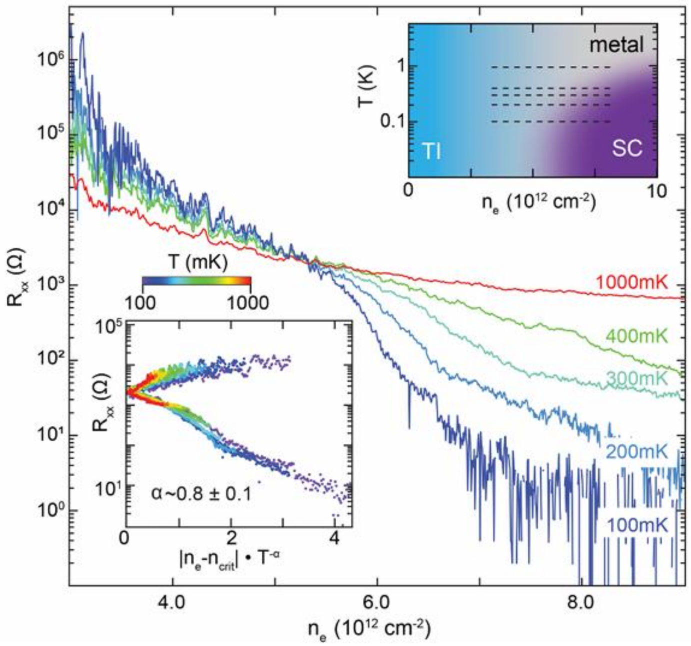
- two key parameters: band gap and interaction strength
- strong interaction:
  hybridized exciton-Cooper pair, phase separation and BEC
- large band gap:
  exciton mediated attraction between conduction electrons
- bonus: unconventional pairing symmetry



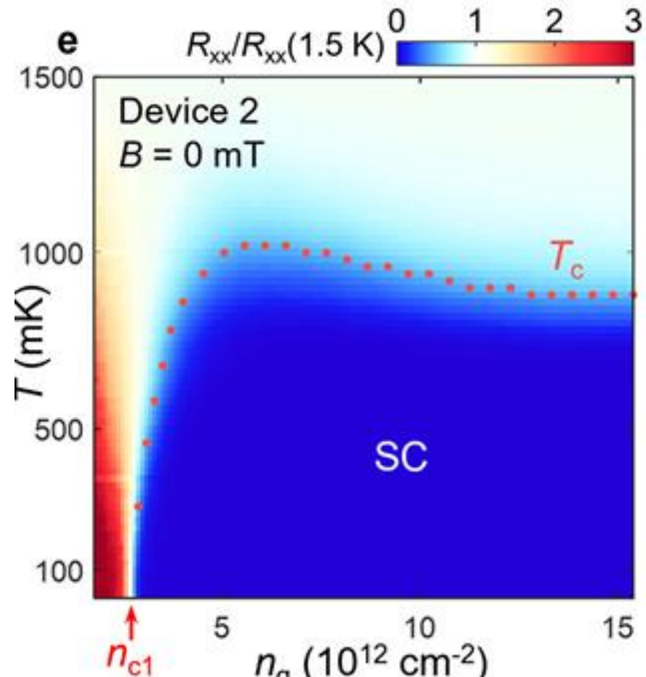


# SC in 2D Materials

Gate-induced SC in band insulators: monolayer $WTe_2$



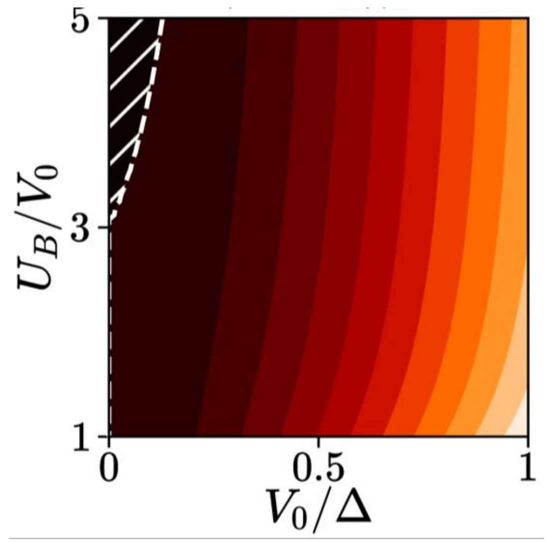

Sajadi et al, Science (2018)
Fatemi et al, Science (2018)

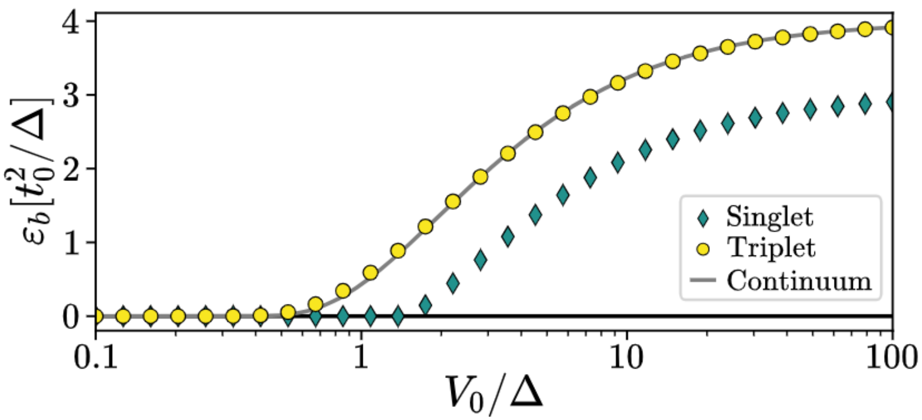



Sanfeng Wu et al, 2501.16699

# f-Wave Spin-Triplet (Valley-Singlet) Pairing

Extended Hubbard model with $U$ and $V$ on honeycomb lattice:




Continuum model for $\nu = 2 + \delta$: Fermi gas with spin and valley

$$\tilde{\mathcal{H}}_i = \int \mathrm{d}x \; g_0(\rho_{K\uparrow}\rho_{K\downarrow} + \rho_{K'\uparrow}\rho_{K'\downarrow}) + g_1 \rho_K \rho_{K'} + g_2 \mathbf{s}_K \cdot \mathbf{s}_{K'}$$

Crepel & LF, PNAS (2022)

# Theory of Spin-Triplet SC in $WTe_2$

- Spin-triplet, valley-singlet pairing parameterized by $d$-vector

$$\psi_{\boldsymbol{d}}(r,r';s,s';v,v') = \psi(r,r')|KK' - K'K\rangle \otimes |\boldsymbol{d}\rangle$$

with $\psi(r,r') = \psi(r',r)$ and $\boldsymbol{d} \cdot (\hat{\boldsymbol{s}}_1 + \hat{\boldsymbol{s}}_2)|\boldsymbol{d}\rangle = 0$

$$|\boldsymbol{Z}\rangle = |\uparrow\downarrow+\downarrow\uparrow\rangle, |\boldsymbol{X}\rangle = |\rightarrow\leftarrow+\leftarrow\rightarrow\rangle = |\uparrow\uparrow+\downarrow\downarrow\rangle, |\boldsymbol{Y}\rangle = i|\uparrow\uparrow-\downarrow\downarrow\rangle.$$

General order parameter: $\boldsymbol{\Delta} \cdot \psi_{\boldsymbol{d}}$ with <u>complex vector</u> $\boldsymbol{\Delta}$

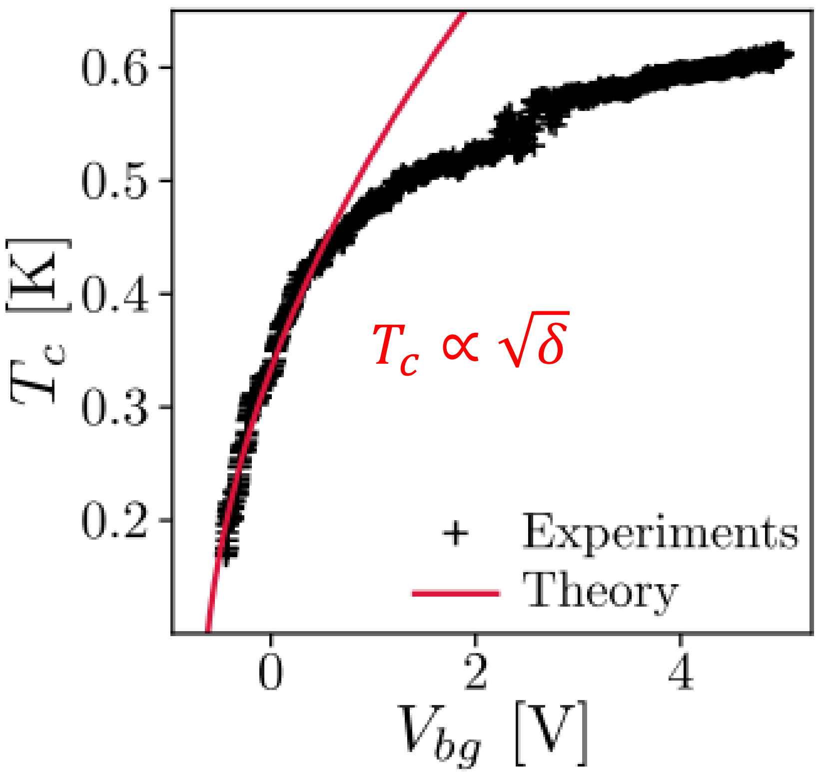
- Zeeman effect on triplet pairs

Landau free energy: $\mathrm{F} = \mathrm{r}(\boldsymbol{\Delta}^* \cdot \boldsymbol{\Delta}) + \mathrm{g}\mathbf{B} \cdot (i\boldsymbol{\Delta} \times \boldsymbol{\Delta}^*) + \boldsymbol{o}(B^2)$

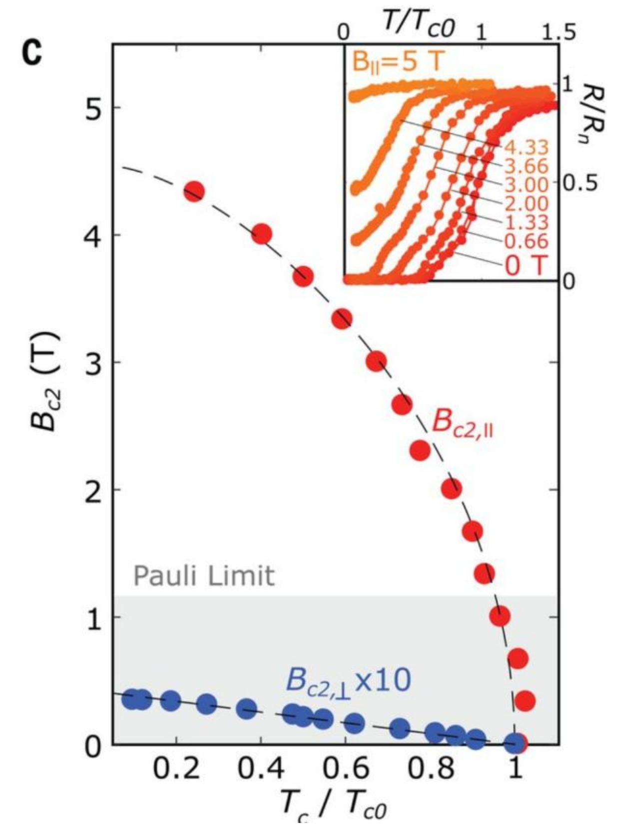
$\mathrm{B}_z(N_\uparrow - N_\downarrow) \rightarrow \mathrm{B}_z(|\Delta_+|^2 - |\Delta_-|^2)$ with $\Delta_\pm = \Delta_x \pm i\Delta_y$

Crepel & LF, PNAS (2022)

# Application to $WTe_2$

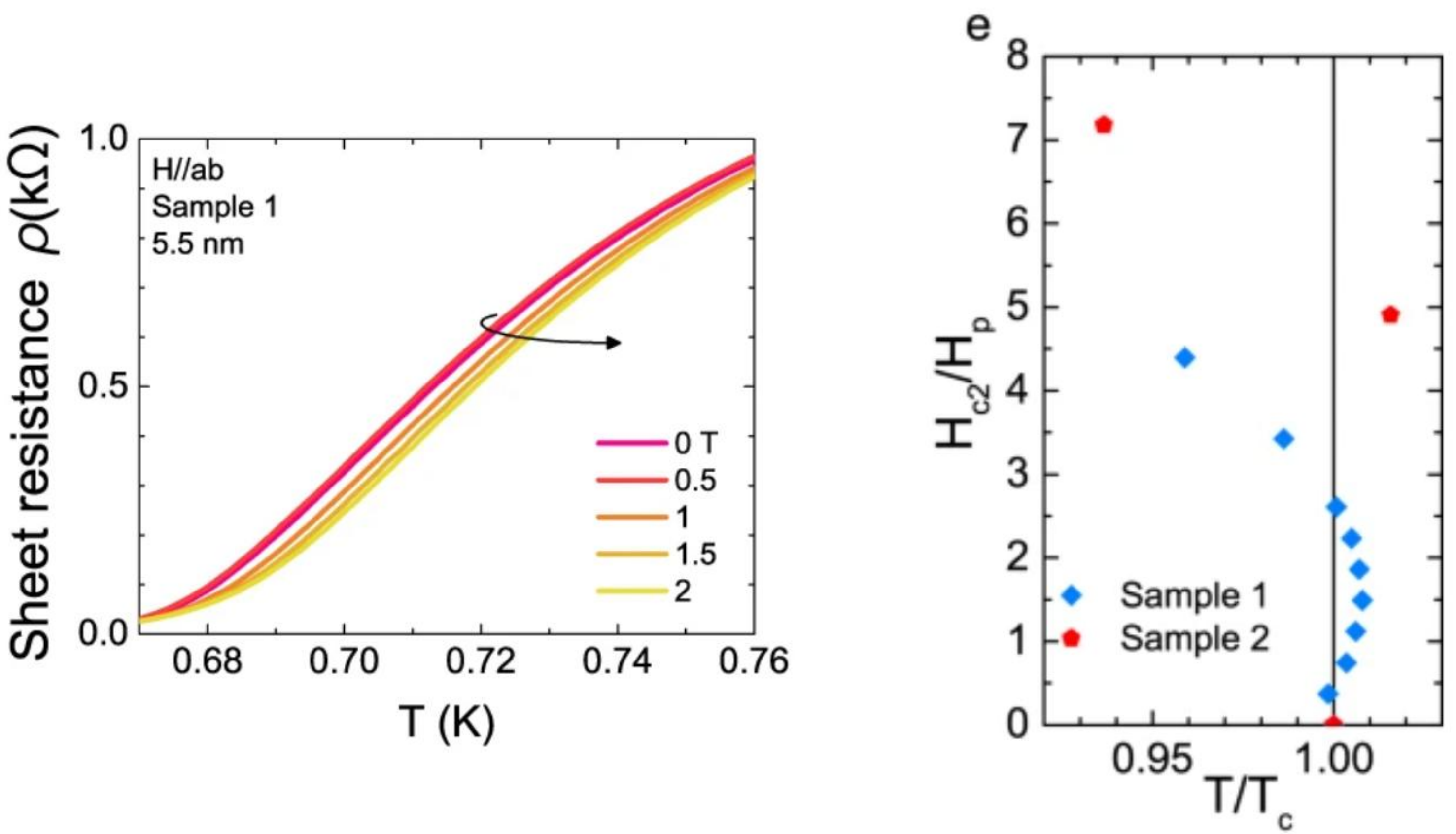



large violation of Pauli limit



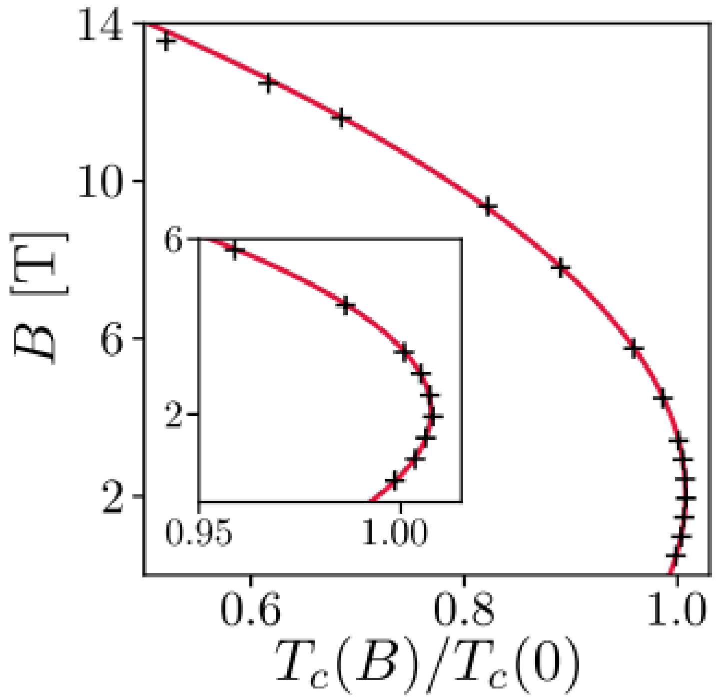

Crepel & LF, PNAS (2022)

# Magnetic Field Enhances Tc!




Lu Li, Sci. Rep. (2018)

# Magnetic Field Enhances Tc!




$$\Delta T_c^B = \mu B - \chi B^2$$

# Search for Spin-Triplet Superconductor

Lesson from $Sr_2RuO_4$ :

Letter | Published: 17 December 1998

**Spin-triplet superconductivity in $Sr_2RuO_4$ identified by $^{17}O$ Knight shift**

Letter | Published: 23 September 2019

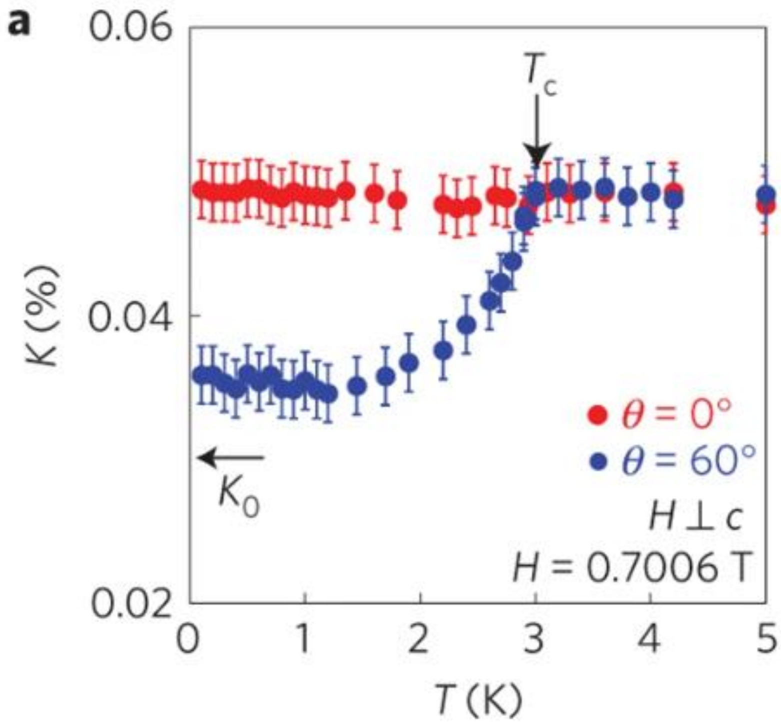
**Constraints on the superconducting order parameter in $Sr_2RuO_4$ from oxygen-17 nuclear magnetic resonance**

Perspective | Published: 11 November 2024

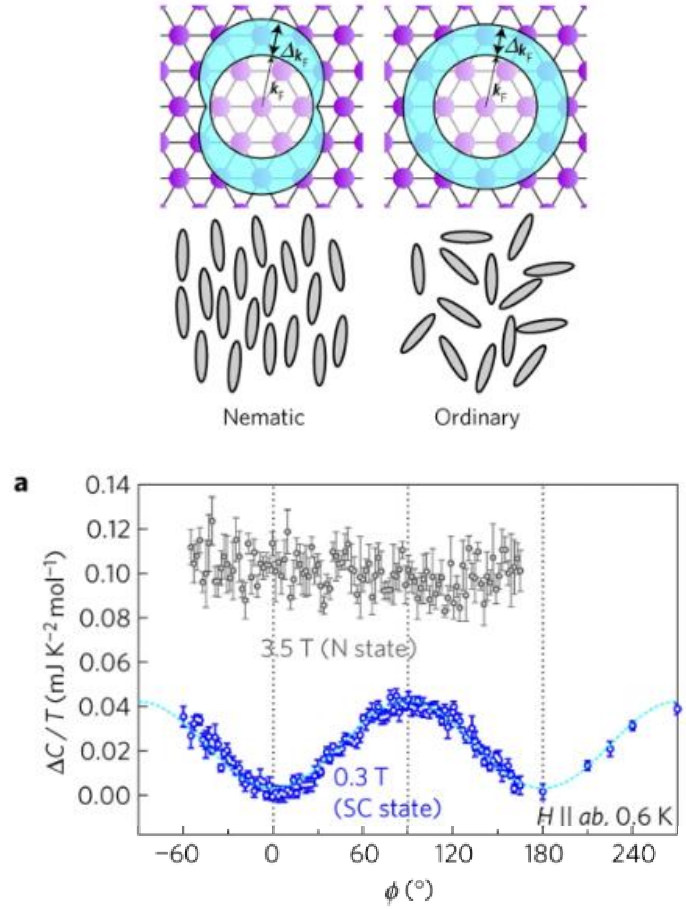
**Thirty years of puzzling superconductivity in $Sr_2RuO_4$**

# Spin-Triplet Nematic Superconductor

$Cu_xBi_2Se_3$




Matano … Zheng, Nat. Phys. (2016)




Yonezawa … Maeno, Nat. Phys. (2017)

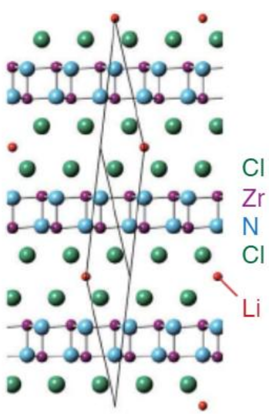
# SC in 2D Materials

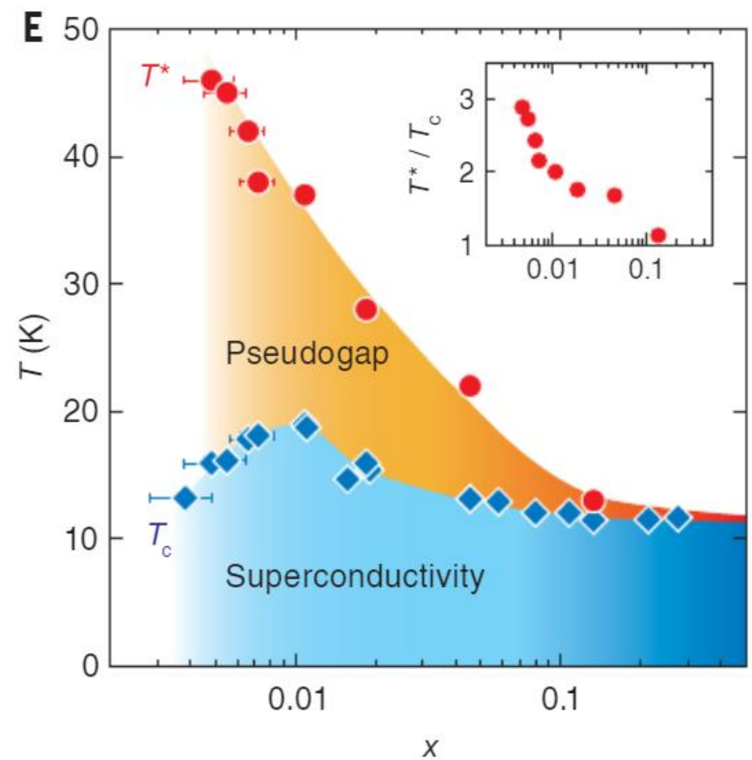
Gate-induced SC in band insulators:  ZrNCl (band gap 2.5eV)

SUPERCONDUCTIVITY

## Gate-controlled BCS-BEC crossover in a two-dimensional superconductor

Yuji Nakagawa[1,2], Yuichi Kasahara[3], Takuya Nomoto[1], Ryotaro Arita[1,4], Tsutomu Nojima[5], Yoshihiro Iwasa[1,2,4]*

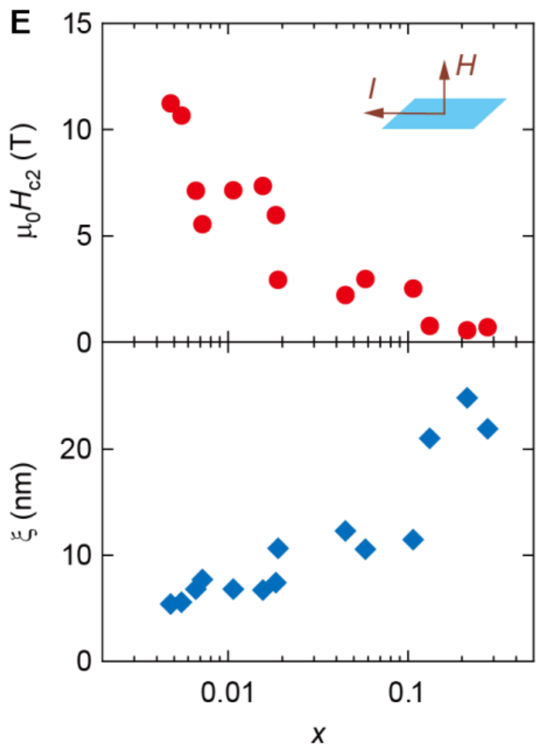


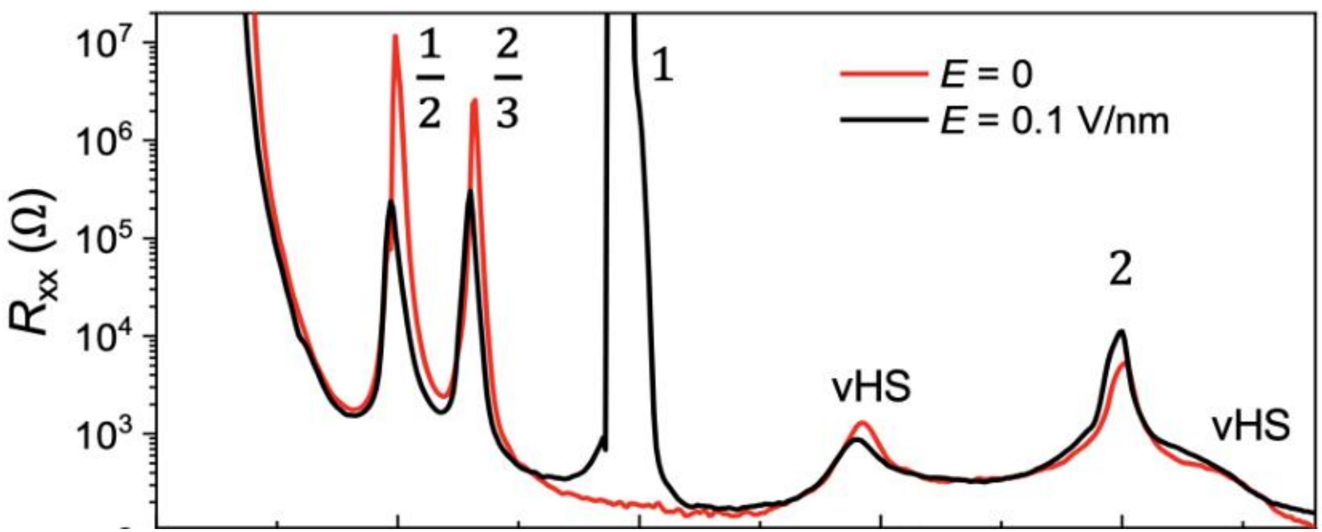



# Non-BCS SC

At low doping: $\Delta \sim \sqrt{E_F \epsilon_b}$ $\quad$ $\mathrm{v_F} \sim \sqrt{E_F/m}$

Coherence length $\xi = \mathrm{v_F}/\Delta$ is independent of doping!

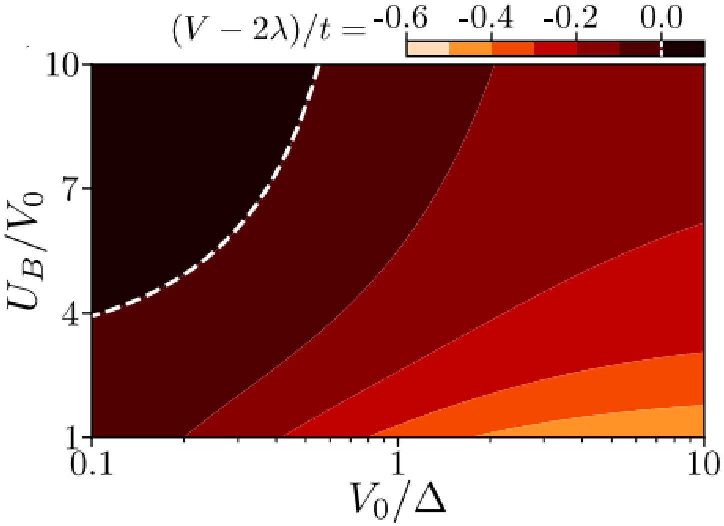


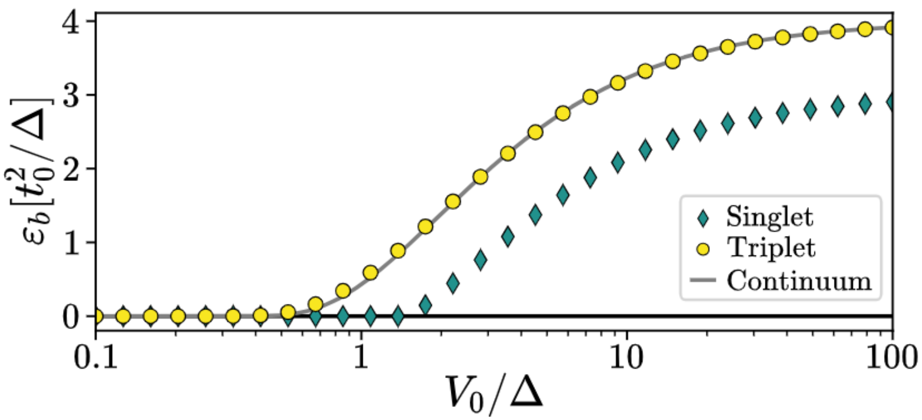


$R_{xx}$ (Ω)
$10^7$
$10^6$
$10^5$
$10^4$
$10^3$
$\frac{1}{2}$
$\frac{2}{3}$
1
2
vHS
vHS
$E$ = 0
$E$ = 0.1 V/nm

# f-Wave Spin-Triplet & Valley-Singlet Pairing

Spinful Hubbard model with $U$ and $V$



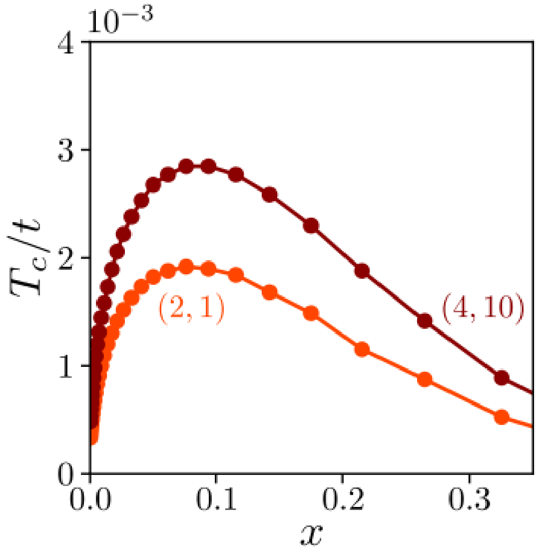

Continuum model: spin + valley

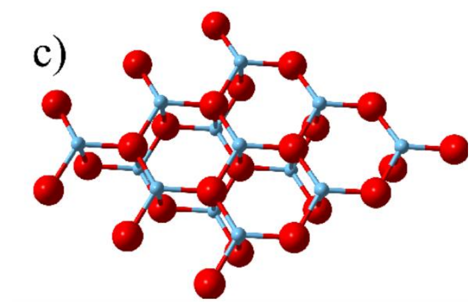
$$\tilde{\mathcal{H}}_i = \int \mathrm{d}x \; g_0(\rho_{K\uparrow}\rho_{K\downarrow} + \rho_{K'\uparrow}\rho_{K'\downarrow}) + g_1\rho_K\rho_{K'} + g_2\mathbf{s}_K \cdot \mathbf{s}_{K'}$$

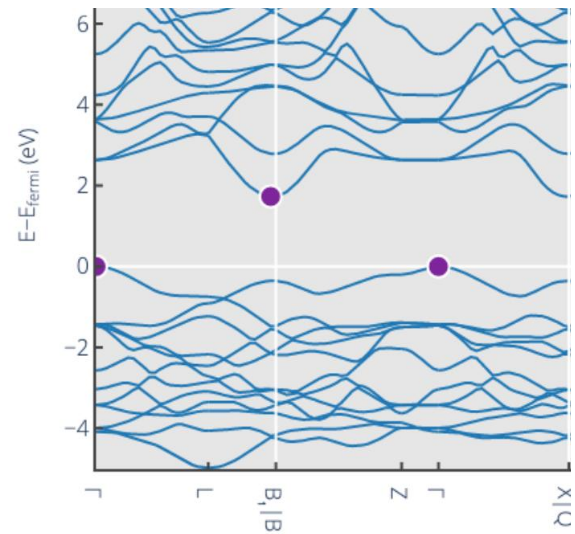
Crepel & LF, PNAS (2022)

# Application to ZrNCl

Crepel & LF, PNAS (2022)



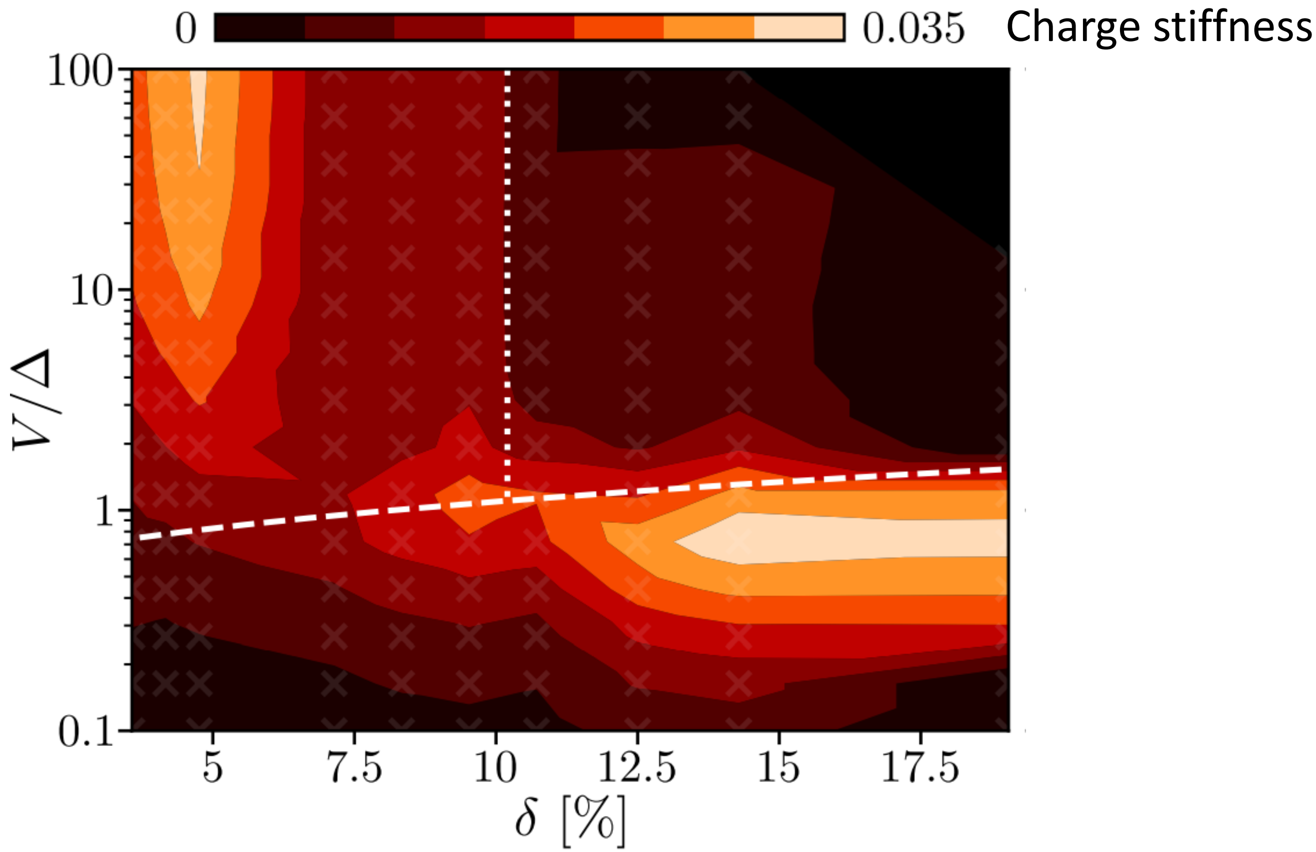




the effective mass at the $K$ point $m = 0.580m_e$ [51, 52], we estimate $t \sim 0.65\,\mathrm{eV}$. With the parameters $U_B = 4V_0 = 4\Delta$ of Fig. 4, we obtain a critical temperature $T_c \sim 0.002t \simeq 15\,\mathrm{K}$ for 9% doping, which lies very close to the experimentally measured value.

Prediction: ZrNCl at low electron doping is a f-wave spin-triplet SC.

# Phase Separation at $V \gg \Delta$



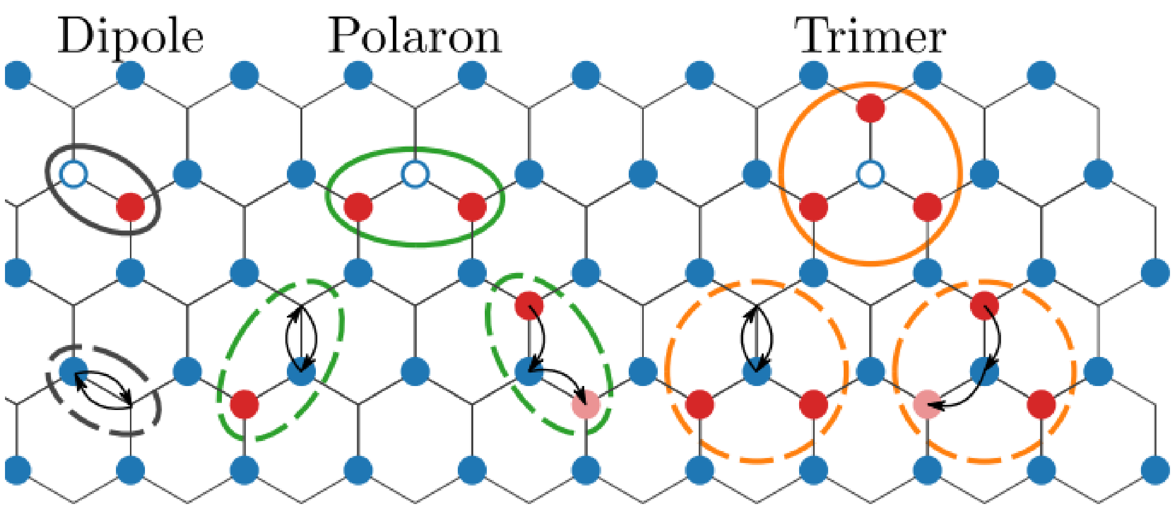
# Equal-Spin Pairing

$\psi(x, y) c_{x\uparrow}^{+} c_{y\uparrow}^{+}$ with $\psi(x, y) = -\psi(y, x)$

# Strong-Coupling Expansion $t \ll \Delta$

$$\mathcal{H}_0 = V \sum_{\langle r,r' \rangle} n_r n_{r'} + \Delta \sum_{r \in B} n_r, \qquad \mathcal{H}_t = -t \sum_{\langle r,r' \rangle} (c_r^\dagger c_{r'} + hc).$$

- hopping of electron from A to B site increases energy.
- low-energy physics governed by second-order processes




- intermediate states: dipoles, polarons and **trimers** cost energies $E_d = \Delta + 2V$, $E_p = \Delta + V$, $\boldsymbol{E_t = \Delta}$ respectively.

# Spin-Polarized Superconductors

- Full spin polarization (lecture 1 & 2)
  - rhombohedral graphene
  - pairing mechanism
  - topology & Majorana

- Partial spin polarization (lecture 3)
  - finite momentum pairing
  - beyond pairing

# Chiral Superconductivity, Pair Density Wave and Majorana Fermions in Rhombohedral Graphene

Liang Fu

SIMONS FOUNDATION

# Collaboration

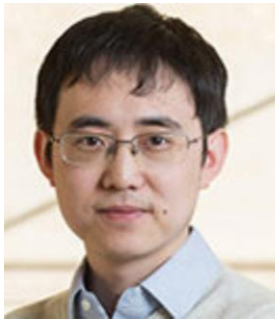
Max Geier

Margarita Davydova
(=>Caltech)

Filippo Gaggioli

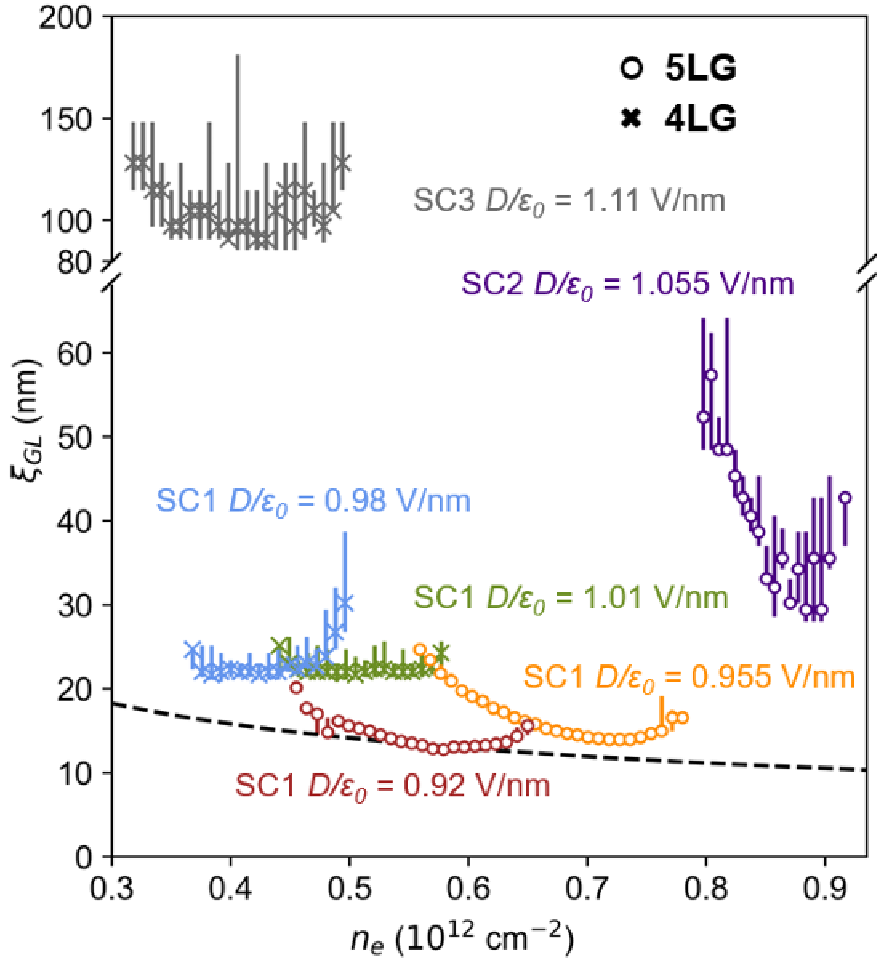
Daniele Guerci

Experiment:



Long Ju

Tonghang Han



200
150
100
80
60
50
40
30
20
10
0
5LG
4LG
SC3 D/ε0 = 1.11 V/nm
SC2 D/ε0 = 1.055 V/nm
SC1 D/ε0 = 0.98 V/nm
SC1 D/ε0 = 1.01 V/nm
SC1 D/ε0 = 0.955 V/nm
SC1 D/ε0 = 0.92 V/nm
ξGL (nm)
0.3
0.4
0.5
0.6
0.7
0.8
0.9
ne (10^12 cm^−2)

# Pairing from Repulsion

**Electronic Correlation Effects and Superconductivity in Doped Fullerenes**

SUDIP CHAKRAVARTY, MARTIN P. GELFAND, STEVEN KIVELSON

**Pair binding in small Hubbard-model molecules**

Steven R. White
*Department of Physics, University of California at Irvine, Irvine, California 92717*

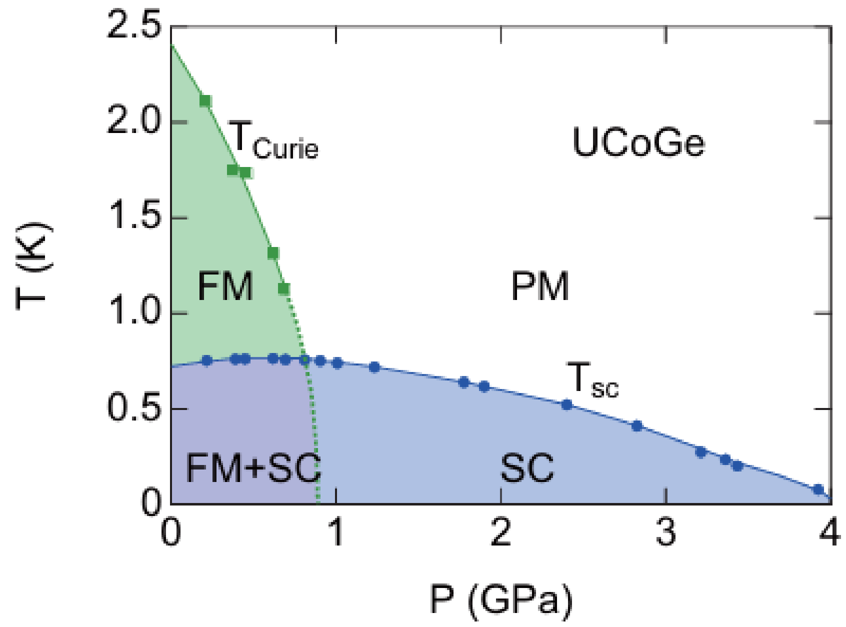
Sudip Chakravarty, Martin P. Gelfand, and Steven A. Kivelson
*Department of Physics, University of California at Los Angeles, Los Angeles, California 90024*

**Optimal inhomogeneity for superconductivity: Finite-size studies**

Wei-Feng Tsai,[1,2] Hong Yao,[2] Andreas Läuchli,[3] and Steven A. Kivelson[2]

Pair Binding Energy: $\Delta = 2E_0(N+1) - E_0(N+2) - E_0(N)$

# Spin-Polarized Superconductivity



# Pairing Competes with Magnetization

Spin-singlet superconductors have zero total spin and exhibit a finite energy gap to spin-flip excitations. Increasing Zeeman field drives a superconductor-to-normal transition at $\mu_0 B_P \approx 1.86 T_c$.

# Spin-Polarized Superconductivity

- Topology & Majorana Fermion
- Rhombohedral Graphene
- Magnonic Cooper pair

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# SC in Graphene Systems