# Bott periodicity for the topological classification of gapped states of matter with reflection symmetry

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# Periodic table of topological insulators & SC

Bott periodicity

| AZ class\ $d$ | 0 | 1 | 2 | 3 | 4 | T | P | C |
|---|---|---|---|---|---|---|---|---|
| A | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| AIII | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | 1 |
| AI | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | $+$ | 0 | 0 |
| BDI | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $+$ | $+$ | 1 |
| D | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $+$ | 0 |
| DIII | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | $-$ | $+$ | 1 |
| AII | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $-$ | 0 | 0 |
| CII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $-$ | $-$ | 1 |
| C | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | 0 | $-$ | 0 |
| CI | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $+$ | $-$ | 1 |

A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, PRB **78**, 195125 (2008)

A. Kitaev, AIP Conf. Proc. 1134, **22** (2009)

# Algebraic relation between P,T and R symmetries

AZ symmetry classes + crystalline symmetry ➔ Crystalline topological insulators (TCIs)

Liang Fu, Phys. Rev. Lett. 106, 106802 (2011)

"effective" symmtries $H_d(\mathbf{k}) = -U_{\mathcal{P}}^\dagger H_d(-\mathbf{k})^* U_{\mathcal{P}}$ $\quad H_d(\mathbf{k}) = U_{\mathcal{T}}^\dagger H_d(-\mathbf{k})^* U_{\mathcal{T}}$

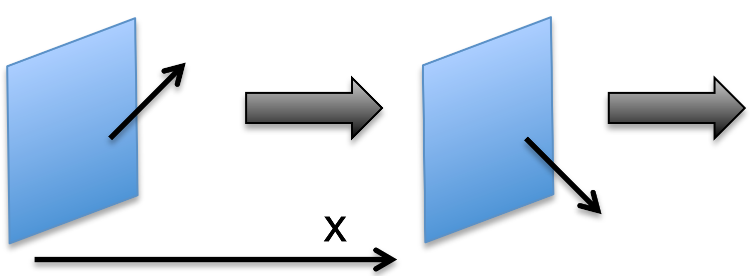
spin-1/2: $\mathcal{T} = \sigma_2 K,\ \mathcal{T}^2 = -1$ + unitary symmetry $\{\mathcal{T}, U_{\mathcal{X}}\} = 0$

"effective" T $\tilde{\mathcal{T}}^2 = (\mathcal{T} U_{\mathcal{X}})^2 = 1$

Ambiguity of algebraic relation between unitary & antiunitary symmetries

$$\mathcal{T} R \mathcal{T}^\dagger = R \quad \Longrightarrow \quad \mathcal{T}(iR)\mathcal{T}^\dagger = -(iR) \qquad R^2 = 1$$

Anticommuting example R, x -> -x:




$$R = \sigma_x \quad \{\mathcal{T}, R\} = 0$$

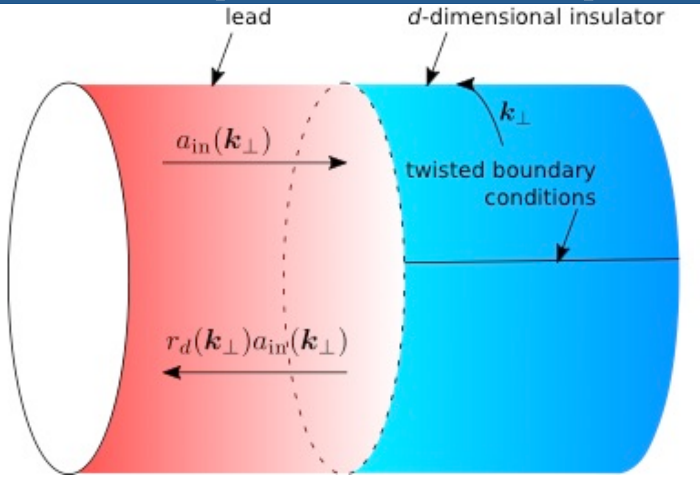
# Big table for TCIs with reflection symmetry

2 or 4 subclasses

| AZ class | T | P | C | | MSC | $d=1$ | $d=2$ | $d=3$ | $d=4$ | $d=5$ | $d=6$ | $d=7$ | $d=8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AIII | 0 | 0 | 1 | $R_+$ | $\mathrm{AIII}^2$ | 0 | $M\mathbb{Z}$ | 0 | $M\mathbb{Z}$ | 0 | $M\mathbb{Z}$ | 0 | $M\mathbb{Z}$ |
| | | | | $R_-$ | A | $\mathbb{Z}^1$ | 0 | $\mathbb{Z}^1$ | 0 | $\mathbb{Z}^1$ | 0 | $\mathbb{Z}^1$ | 0 |
| A | 0 | 0 | 0 | $R$ | $\mathrm{A}^2$ | $M\mathbb{Z}$ | 0 | $M\mathbb{Z}$ | 0 | $M\mathbb{Z}$ | 0 | $M\mathbb{Z}$ | 0 |
| AI | + | 0 | 0 | $R_+$[a] | $\mathrm{AI}^2$ | $M\mathbb{Z}$ | 0 | 0 | 0 | $2M\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| | | | | $R_-$ | A | 0 | 0 | $2M\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 |
| BDI | + | + | 1 | $R_{++}$[a] | $\mathrm{BDI}^2$ | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 | 0 | $2M\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| | | | | $R_{--}$ | AIII | 0 | 0 | 0 | $2M\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}_2$ | $M\mathbb{Z}$ |
| | | | | $R_{+-}$ | AI | $2\mathbb{Z}^1$ | 0 | 0 | 0 | $\mathbb{Z}^1$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| | | | | $R_{-+}$ | D | $2\mathbb{Z}$ | 0 | $2M\mathbb{Z}$ | 0 | $2\mathbb{Z}$ | 0 | $2M\mathbb{Z}$ | 0 |
| D | 0 | + | 0 | $R_+$[a] | $\mathrm{D}^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 | 0 | $2M\mathbb{Z}$ | 0 |
| | | | | $R_-$[b] | A | $M\mathbb{Z}$ | 0 | 0 | 0 | $2M\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}_2$ |
| DIII | − | + | 1 | $R_{++}$ | $\mathrm{DIII}^2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 | 0 | $2M\mathbb{Z}$ |
| | | | | $R_{--}$[b] | AIII | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 | 0 | $2M\mathbb{Z}$ | 0 | 0 |
| | | | | $R_{+-}$ | AII | $2M\mathbb{Z}$ | 0 | $2\mathbb{Z}$ | 0 | $2M\mathbb{Z}$ | 0 | 0 | $2\mathbb{Z}$ |
| | | | | $R_{-+}$ | D | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}^1$ | 0 | 0 | 0 | $2\mathbb{Z}^1$ | 0 |
| AII | − | 0 | 0 | $R_+$ | $\mathrm{AII}^2$ | $2M\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 | 0 |
| | | | | $R_-$[b] | A | 0 | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 | 0 | $2M\mathbb{Z}$ | 0 |
| CII | − | − | 1 | $R_{++}$ | $\mathrm{CII}^2$ | 0 | $2M\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 |
| | | | | $R_{--}$ | AIII | 0 | 0 | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 | 0 | $2M\mathbb{Z}$ |
| | | | | $R_{+-}$ | AII | $2\mathbb{Z}^1$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}^1$ | 0 | 0 | 0 |
| | | | | $R_{-+}$ | C | $2\mathbb{Z}$ | 0 | $2M\mathbb{Z}$ | 0 | $2\mathbb{Z}$ | 0 | $2M\mathbb{Z}$ | 0 |
| C | 0 | − | 0 | $R_+$[c] | $\mathrm{C}^2$ | 0 | 0 | $2M\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 |
| | | | | $R_-$ | A | $2M\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 | 0 |
| CI | + | − | 1 | $R_{++}$[d] | $\mathrm{CI}^2$ | 0 | 0 | 0 | $2M\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $M\mathbb{Z}$ |
| | | | | $R_{--}$ | AIII | 0 | $2M\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}_2$ | $M\mathbb{Z}$ | 0 | 0 |
| | | | | $R_{+-}$ | AI | $2M\mathbb{Z}$ | 0 | $2\mathbb{Z}$ | 0 | $2M\mathbb{Z}$ | 0 | 0 | $2\mathbb{Z}$ |
| | | | | $R_{-+}$ | C | 0 | 0 | $2\mathbb{Z}^1$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}^1$ | 0 |

C.-K. Chiu, H. Yao, and S. Ryu, Phys. Rev. B 88, 075142 (2013)

# Bott periodicity from the scattering matrix




Dimensional reduction

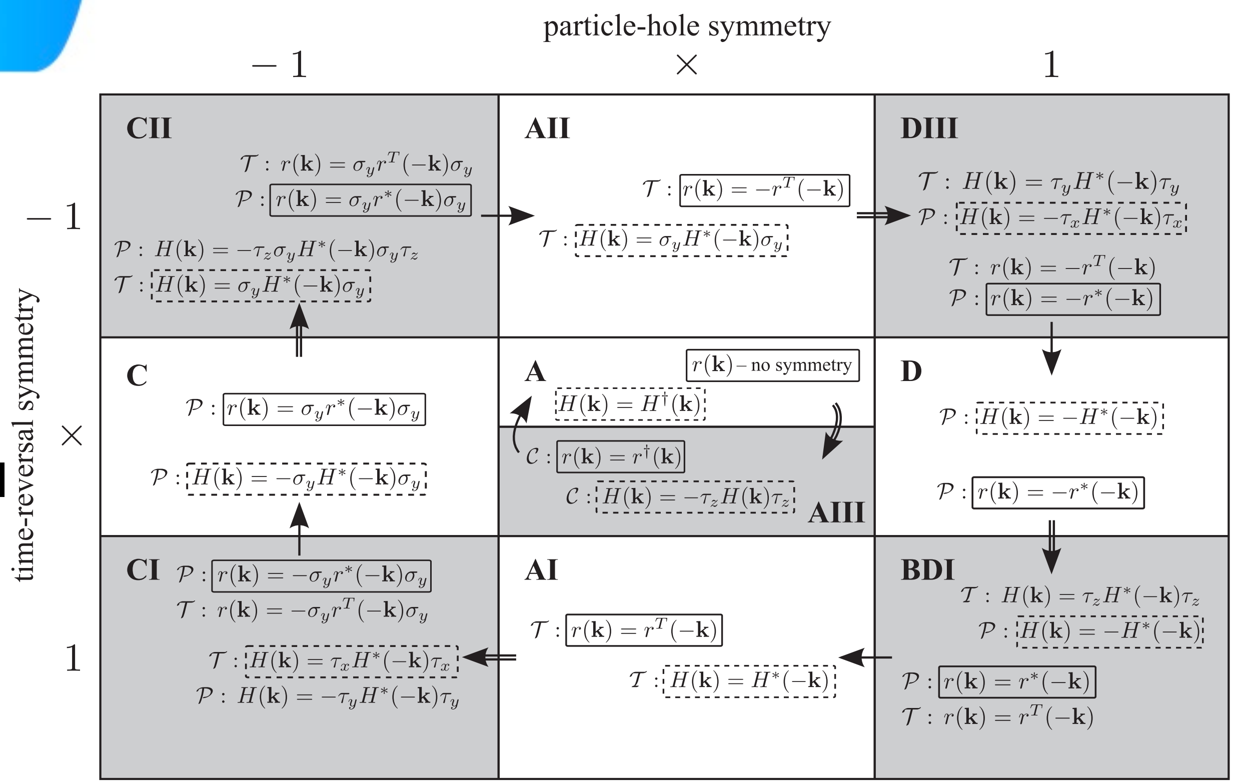
$$H_{d-1}(\boldsymbol{k}) \equiv r(\boldsymbol{k}), \quad \text{with chiral symmetry,} \qquad (3.4a)$$

$$H_{d-1}(\boldsymbol{k}) \equiv \begin{pmatrix} 0 & r(\boldsymbol{k}) \\ r^\dagger(\boldsymbol{k}) & 0 \end{pmatrix}, \quad \text{without chiral symmetry.}$$

Changes the symmetry class

BUT

$H_{d-1}$ has the same topological invariants




I. C. Fulga, F. Hassler, and A. R. Akhmerov, Phys. Rev. B 85, 165409 (2012)

# Bott periodicity for TCIs

Antiunitary symmetries $H_d(\mathbf{k}) = -U_{\mathcal{P}}^{\dagger} H_d(-\mathbf{k})^* U_{\mathcal{P}}$ $H_d(\mathbf{k}) = U_{\mathcal{T}}^{\dagger} H_d(-\mathbf{k})^* U_{\mathcal{T}}$

Unitary symmetries $H_d(\mathbf{k}) = -U_{\mathcal{C}}^{\dagger} H_d(\mathbf{k}) U_{\mathcal{C}}$ $H_d(\mathbf{k}) = U_{\mathcal{R}}^{\dagger} H_d(R\mathbf{k}) U_{\mathcal{R}}$

Leads need to be attached in a reflection symmetric way
➔dimensional reduction down to d=1

Bott periodicity for complex AZ classes
2 period-two sequences

$$\mathrm{A}^{\mathcal{R}} \xrightarrow{d-1} \mathrm{AIII}^{\mathcal{R}_+} \xrightarrow{d-1} \mathrm{A}^{\mathcal{R}}$$

$$\mathrm{A}^{\mathcal{C}\mathcal{R}} \xrightarrow{d-1} \mathrm{AIII}^{\mathcal{R}_-} \xrightarrow{d-1} \mathrm{A}^{\mathcal{C}\mathcal{R}}$$

new subclass, C (chiral) and R (reflection) symmetries broken, CR still the symmetry

real AZ classes 4 period-eight sequences

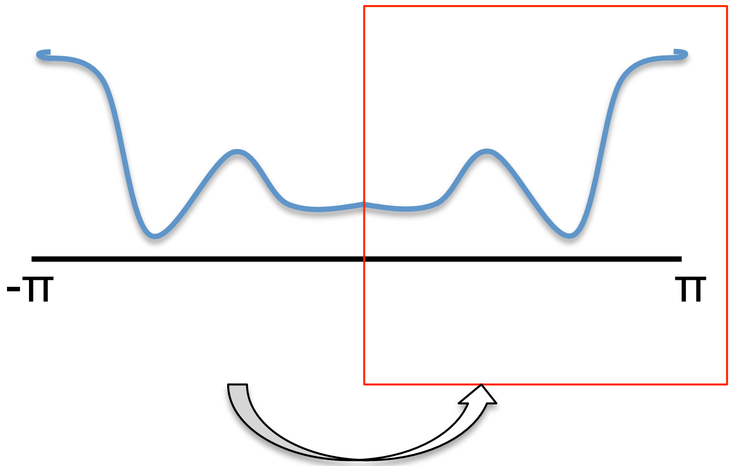
$$\mathrm{CI}^{\mathcal{R}_{++}} \xrightarrow{d-1} \mathrm{C}^{\mathcal{R}_+} \xrightarrow{d-1} \mathrm{CII}^{\mathcal{R}_{++}} \xrightarrow{d-1} \mathrm{AII}^{\mathcal{R}_+} \xrightarrow{d-1} \mathrm{DIII}^{\mathcal{R}_{++}} \xrightarrow{d-1} \mathrm{D}^{\mathcal{R}_+} \xrightarrow{d-1} \mathrm{BDI}^{\mathcal{R}_{++}} \xrightarrow{d-1} \mathrm{AI}^{\mathcal{R}_+} \xrightarrow{d-1} \mathrm{CI}^{\mathcal{R}_{++}}$$

$$\mathrm{CI}^{\mathcal{R}_{--}} \xrightarrow{d-1} \mathrm{C}^{\mathcal{R}_-} \xrightarrow{d-1} \mathrm{CII}^{\mathcal{R}_{--}} \xrightarrow{d-1} \mathrm{AII}^{\mathcal{R}_-} \xrightarrow{d-1} \mathrm{DIII}^{\mathcal{R}_{--}} \xrightarrow{d-1} \mathrm{D}^{\mathcal{R}_-} \xrightarrow{d-1} \mathrm{BDI}^{\mathcal{R}_{--}} \xrightarrow{d-1} \mathrm{AI}^{\mathcal{R}_-} \xrightarrow{d-1} \mathrm{CI}^{\mathcal{R}_{--}}$$

$$\mathrm{CI}^{\mathcal{R}_{-+}} \xrightarrow{d-1} \mathrm{C}^{\mathcal{C}\mathcal{R}_+} \xrightarrow{d-1} \mathrm{CII}^{\mathcal{R}_{+-}} \xrightarrow{d-1} \mathrm{AII}^{\mathcal{C}\mathcal{R}_-} \xrightarrow{d-1} \mathrm{DIII}^{\mathcal{R}_{-+}} \xrightarrow{d-1} \mathrm{D}^{\mathcal{C}\mathcal{R}_+} \xrightarrow{d-1} \mathrm{BDI}^{\mathcal{R}_{+-}} \xrightarrow{d-1} \mathrm{AI}^{\mathcal{C}\mathcal{R}_-} \xrightarrow{d-1} \mathrm{CI}^{\mathcal{R}_{-}}$$

$$\mathrm{CI}^{\mathcal{R}_{+-}} \xrightarrow{d-1} \mathrm{C}^{\mathcal{C}\mathcal{R}_-} \xrightarrow{d-1} \mathrm{CII}^{\mathcal{R}_{-+}} \xrightarrow{d-1} \mathrm{AII}^{\mathcal{C}\mathcal{R}_+} \xrightarrow{d-1} \mathrm{DIII}^{\mathcal{R}_{+-}} \xrightarrow{d-1} \mathrm{D}^{\mathcal{C}\mathcal{R}_-} \xrightarrow{d-1} \mathrm{BDI}^{\mathcal{R}_{-+}} \xrightarrow{d-1} \mathrm{AI}^{\mathcal{C}\mathcal{R}_+} \xrightarrow{d-1} \mathrm{CI}^{\mathcal{R}_{+}}$$

# Topological classification with R-symmetry




R-symmetry

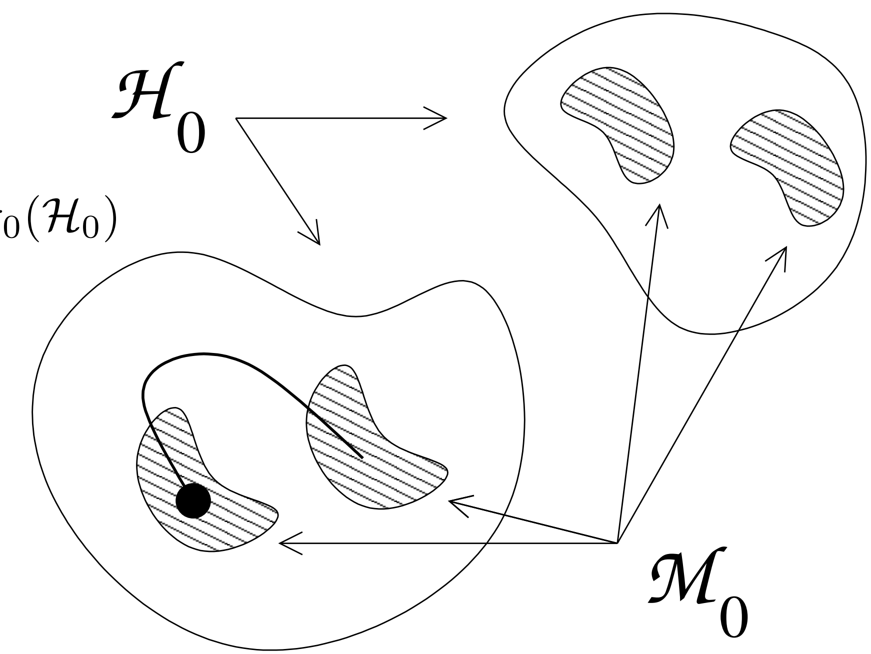
Hamiltonians defined on half of the BZ

at arbitrary k-point H in $H_0$
at mirror planes H in $M_0$

➔ relative homotopy group $\pi_1(\mathcal{H}_0, \mathcal{M}_0)$.

Exact sequence (d=1)

$$\pi_1(\mathcal{M}_0) \stackrel{i_1}{\hookrightarrow} \pi_1(\mathcal{H}_0) \stackrel{j_1}{\hookrightarrow} \pi_1(\mathcal{H}_0, \mathcal{M}_0) \stackrel{\delta}{\to} \pi_0(\mathcal{M}_0) \stackrel{i_0}{\hookrightarrow} \pi_0(\mathcal{H}_0)$$




generators:

$$p = j_1(l) + \delta^{-1} r$$

🡇

topological classification,
definition of topological indices
generators (Hamiltonians)

# Results of the classification

complex AZ classes

| class | $\mathcal{H}_0^i$ | $\mathcal{R}_0^i$ | $d=1$ | $d=2$ | $d=3$ | $d=4$ |
|---|---|---|---|---|---|---|
| $\mathrm{A}^{\mathcal{R}}$ | AI | $\mathrm{AI}^2$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| $\mathrm{AIII}^{\mathcal{R}+}$ | AIII | $\mathrm{AIII}^2$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| $\mathrm{A}^{\mathcal{CR}}$ | A | AIII | 0 | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}^2$ |
| $\mathrm{AIII}^{\mathcal{R}-}$ | AIII | A | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}^2$ | 0 |

second descendant $Z_2$

real AZ classes

| | | | | | | |
|---|---|---|---|---|---|---|
| $\mathrm{AI}^{\mathcal{RC}-}$ | C | CI | 0 | 0 | 0 | $2\mathbb{Z}^2$ |
| $\mathrm{BDI}^{\mathcal{R}+-}$ | CI | AI | $\mathbb{Z}^2$ | 0 | 0 | 0 |
| $\mathrm{D}^{\mathcal{RC}+}$ | AI | BDI | $\mathbb{Z}_2^2$ | $\mathbb{Z}^2$ | 0 | 0 |
| $\mathrm{DIII}^{\mathcal{R}-+}$ | BDI | D | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}^2$ | 0 |
| $\mathrm{AII}^{\mathcal{RC}-}$ | D | DIII | 0 | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}^2$ |
| $\mathrm{CII}^{\mathcal{R}+-}$ | DIII | AII | $2\mathbb{Z}^2$ | 0 | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ |
| $\mathrm{C}^{\mathcal{RC}+}$ | AII | CII | 0 | $2\mathbb{Z}^2$ | 0 | $\mathbb{Z}_2^2$ |
| $\mathrm{CI}^{\mathcal{R}-+}$ | CII | C | 0 | 0 | $2\mathbb{Z}^2$ | 0 |
| $\mathrm{AI}^{\mathcal{RC}+}$ | D | BDI | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| $\mathrm{BDI}^{\mathcal{R}-+}$ | DIII | D | $\mathbb{Z}$ | 0 | $2\mathbb{Z}$ | 0 |
| $\mathrm{D}^{\mathcal{RC}-}$ | AII | DIII | 0 | $\mathbb{Z}$ | 0 | $2\mathbb{Z}$ |
| $\mathrm{DIII}^{\mathcal{R}+-}$ | CII | AII | $2\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| $\mathrm{AII}^{\mathcal{RC}+}$ | C | CII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| $\mathrm{CII}^{\mathcal{R}-+}$ | CI | C | $\mathbb{Z}$ | 0 | $2\mathbb{Z}$ | 0 |
| $\mathrm{C}^{\mathcal{RC}-}$ | AI | CI | 0 | $\mathbb{Z}$ | 0 | $2\mathbb{Z}$ |
| $\mathrm{CI}^{\mathcal{R}+-}$ | BDI | AI | $2\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |

| | | | | | | |
|---|---|---|---|---|---|---|
| $\mathrm{AI}^{\mathcal{R}-}$ | AII | A | 0 | 0 | $2\mathbb{Z}$ | 0 |
| $\mathrm{BDI}^{\mathcal{R}--}$ | CII | AIII | 0 | 0 | 0 | $2\mathbb{Z}$ |
| $\mathrm{D}^{\mathcal{R}-}$ | C | A | $\mathbb{Z}$ | 0 | 0 | 0 |
| $\mathrm{DIII}^{\mathcal{R}--}$ | CI | AIII | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| $\mathrm{AII}^{\mathcal{R}-}$ | AI | A | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| $\mathrm{CII}^{\mathcal{R}--}$ | BDI | AIII | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| $\mathrm{C}^{\mathcal{R}-}$ | D | A | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\mathrm{CI}^{\mathcal{R}--}$ | DIII | AIII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| class | $\mathcal{H}_0^i$ | $\mathcal{R}_0^i$ | $d=1$ | $d=2$ | $d=3$ | $d=4$ |
| $\mathrm{AI}^{\mathcal{R}+}$ | AI | $\mathrm{AI}^2$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| $\mathrm{BDI}^{\mathcal{R}++}$ | BDI | $\mathrm{BDI}^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| $\mathrm{D}^{\mathcal{R}+}$ | D | $\mathrm{D}^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| $\mathrm{DIII}^{\mathcal{R}++}$ | DIII | $\mathrm{DIII}^2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| $\mathrm{AII}^{\mathcal{R}+}$ | AII | $\mathrm{AII}^2$ | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\mathrm{CII}^{\mathcal{R}++}$ | CII | $\mathrm{CII}^2$ | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| $\mathrm{C}^{\mathcal{R}+}$ | C | $\mathrm{C}^2$ | 0 | 0 | $2\mathbb{Z}$ | 0 |
| $\mathrm{CI}^{\mathcal{R}++}$ | CI | $\mathrm{CI}^2$ | 0 | 0 | 0 | $2\mathbb{Z}$ |

# Literature overview

C.-K. Chiu, H. Yao, and S. Ryu, Phys. Rev. B **88**, 075142 (2013)

Minimal Dirac Hamiltonians (algebraic)

T. Morimoto and A. Furusaki, Phys. Rev. B **88**, 125129 (2013)

Clifford algebras (corrected some classification entries)

K. Shiozaki and M. Sato, Phys. Rev. B **90**, 165114 (2014)

Dimensional reduction for the Hamiltonian down to d=0, method by
*J. C. Y. Teo and C. L. Kane, Phys. Rev. B 82, 115120 (2010)*
Clifford algebra classification for d=0

THE END