Slides for Capri School: Surface Codes

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Capri April 24-28 2017

Lecture 1: Introduction

- Qubits
- Operators
- Measurement
- Quantum circuits
- Superconducting qubits
- Error models

(Qubits, Operators, Measurement: "all" of quantum mechanics)

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References on general QM & QC

E D Commins, Quantum Mechanics: An Experimentalist's Approach (Cambridge, 2014) ISBN 978-1107063990

J J Sakurai and J Napolitano, Modern Quantum Mechanics (2nd ed) (Addison-Wesley, 2011) ISBN 978-0-8053-8291-4

R Shankar, Principles of Quantum Mechanics (2nd ed) (Springer, 1994) ISBN 978-1-4757-0576-8

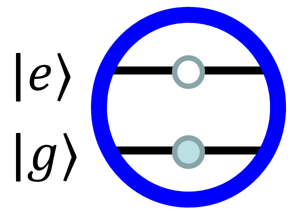
M Nielsen and I Chuang, Quantum Computation and Quantum Information (Cambridge, 2011) ISBN-13: 978-1107002173

J Preskill, http://www.theory.caltech.edu/people/preskill/ph229/ Lecture notes for Quantum Information, Caltech Physics 229

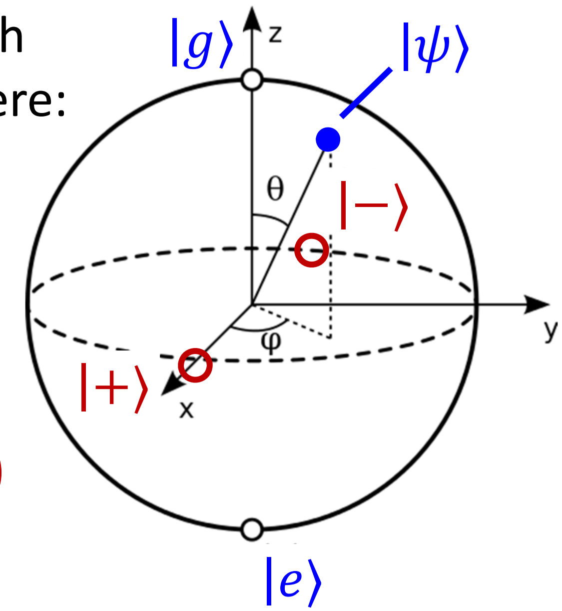
A.G. Fowler et al, *Phys Rev A* **86**, 032324 (2012)

Qubits

Single qubits:



Bloch sphere:



$$|\psi\rangle = a|g\rangle + b|e\rangle$$ "Z" basis

$$|a|^2 + |b|^2 = 1 \quad \begin{cases} a = \cos(\theta/2) \\ b = e^{i\varphi}\sin(\theta/2) \end{cases}$$

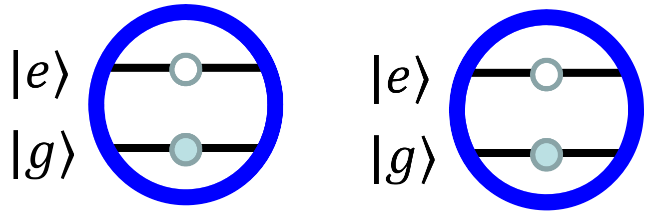
"X" basis:

$$|+\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|g\rangle - |e\rangle)$$

$$|\psi\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$$

Qubits

Two qubits:



Product state:
No entanglement

$$|\psi\rangle = (a|g\rangle + b|e\rangle) \otimes (c|g\rangle + d|e\rangle)$$

$$= ac|gg\rangle + ad|ge\rangle + bc|eg\rangle + bd|ee\rangle$$

qubit 1

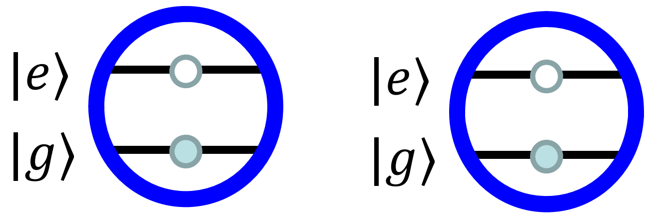
qubit 2

"Z" basis

In general $|\psi\rangle$ cannot be written in product form (entangled states)

Bell Basis

Two qubits:



Bell states provide entangled state basis:

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|gg\rangle + |ee\rangle) \qquad |\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|ge\rangle + |eg\rangle)$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|gg\rangle - |ee\rangle) \qquad |\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|ge\rangle - |eg\rangle)$$

Indices relate to standard quantum circuit used to generate Bell states: will discus more later

Qubits

Four qubits:

$2^4 = 16$ independent coefficients

$$|\psi\rangle = a|gggg\rangle + b|ggge\rangle + c|ggeg\rangle + \cdots$$

N qubits:

$$|\psi\rangle = \sum_{j=0}^{2^N} a_j\,|j\rangle$$

100 qubits:
$2^{100} \approx 1.3 \times 10^{28}$ coefficients

More than world's current digital storage capacity…

Operators

Operator A: $A|u\rangle = |v\rangle$

Dual space "bras"

Adjoint operator $A^\dagger$: $\langle u|A^\dagger = \langle v|$

Unitary operator: $A^\dagger A = AA^\dagger = I$

Hermitian operator: $A^\dagger = A$

Given a set of basis vectors $|n\rangle$ for the Hilbert space :

$$A|n\rangle = a_{0n}|0\rangle + a_{1n}|1\rangle + a_{2n}|2\rangle + \cdots$$

$$\langle m|A|n\rangle = a_{mn}$$ ⬅ Representation of A

$$\langle n|A^\dagger|m\rangle = a_{mn}^\star$$ ⬅ Representation of $A^\dagger$

Representation of $A^\dagger$ = c.c. transpose of A's representation

Hermitian Operators

A equals $A^\dagger$: $$a_{mn} = a^\star_{nm}$$

Eigenvector-eigenvalue equation:

$$A|n\rangle = \lambda_n |n\rangle$$

Eigenvector — Eigenvalues are all real

Eigenvectors provide orthonormal basis; representation in eigenvector basis is diagonal

$$A \Leftrightarrow \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Operator products & commutation

Operator commutator: $[A, B] = AB - BA$

If $[A, B] = 0$ operators "commute"

Anti-commutator: $\{A, B\} = AB + BA$

If $\{A, B\} = 0$ operators "anti-commute"

Note $AB|\psi\rangle$ means operate first with B then with A

Representation of AB is product of matrix representations of A and B

$$AB \Leftrightarrow \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{n0} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{00} & \cdots & b_{0n} \\ \vdots & \ddots & \vdots \\ b_{n0} & \cdots & b_{nn} \end{pmatrix}$$

Single qubit operators

(all in the $|g\rangle, |e\rangle$ basis)

$$X = \sigma_x \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \sigma_y \Leftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \sigma_z \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\{\sigma_j, \sigma_k\} = 0$$

$$\sigma_j^2 = I$$

Any single qubit operator:

$$A = aI + bX + cY + dZ$$

Hadamard $\quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

π rotation about axis halfway between $\hat{x}$ and $\hat{z}$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \qquad T^4 = S^2 = Z$$

"$\pi/8$ gate"

Single qubit operators

Bit-flip operator:

$$X|g\rangle \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow |e\rangle$$

$$X|e\rangle = |g\rangle$$

Phase-flip operator:

$$Z|g\rangle \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow |g\rangle$$

$$Z|e\rangle = -|e\rangle$$

Combined bit & phase flip:

$$Y|g\rangle = iXZ|g\rangle = i|e\rangle$$

$$Y|e\rangle = iXZ|e\rangle = -i|g\rangle$$

Basis change

$$H|g\rangle = (|g\rangle + |e\rangle)/\sqrt{2} = |+\rangle$$

$$H|e\rangle = (|g\rangle - |e\rangle)/\sqrt{2} = |-\rangle$$

Two qubit operators

CNOT gate: if <u>control qubit</u> in $|e\rangle$, flip (NOT) <u>target qubit</u>

Truth table:

$\|q_1\rangle$	$\|q_2\rangle$	$\|q_1 q_2\rangle$
$\|g\rangle$	$\|g\rangle$	$\|gg\rangle$
$\|g\rangle$	$\|e\rangle$	$\|ge\rangle$
$\|e\rangle$	$\|g\rangle$	$\|ee\rangle$
$\|e\rangle$	$\|e\rangle$	$\|eg\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CZ gate: if <u>control qubit</u> in $|e\rangle$, apply Z to <u>target qubit</u>

Truth table:

$\|q_1\rangle$	$\|q_2\rangle$	$\|q_1 q_2\rangle$
$\|g\rangle$	$\|g\rangle$	$\|gg\rangle$
$\|g\rangle$	$\|e\rangle$	$\|ge\rangle$
$\|e\rangle$	$\|g\rangle$	$\|eg\rangle$
$\|e\rangle$	$\|e\rangle$	$-\|ee\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Measurement

Any physical measurement is represented by a Hermitian operator

1. The measurement outcome is an eigenvalue of the operator
2. The state resulting from measurement is the corresponding eigenvector
3. The probability of an outcome is the squared amplitude of that eigenvector in the original state decomposition

$$|\psi\rangle = a|g\rangle + b|e\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle.$$

Z measurement:

+1 eigenvalue: Probability $|a|^2$, Outcome $|g\rangle$ — $M_Z|\psi\rangle = +1$

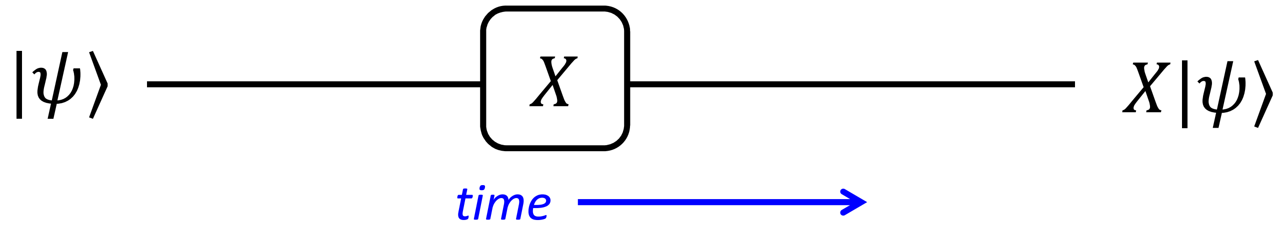
-1 eigenvalue: Probability $|b|^2$, Outcome $|e\rangle$ — $M_Z|\psi\rangle = -1$

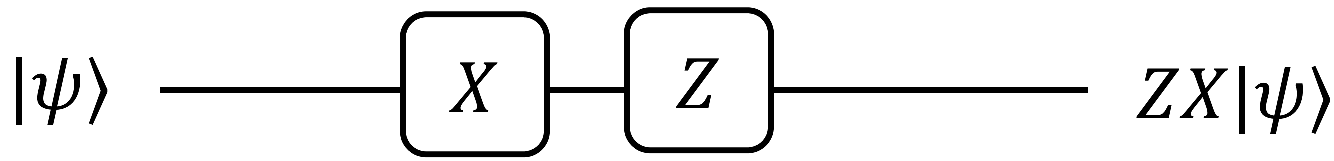
X measurement:

+1 eigenvalue: Probability $|a+b|^2/2$, Outcome $|+\rangle$ — $M_X|\psi\rangle = +1$

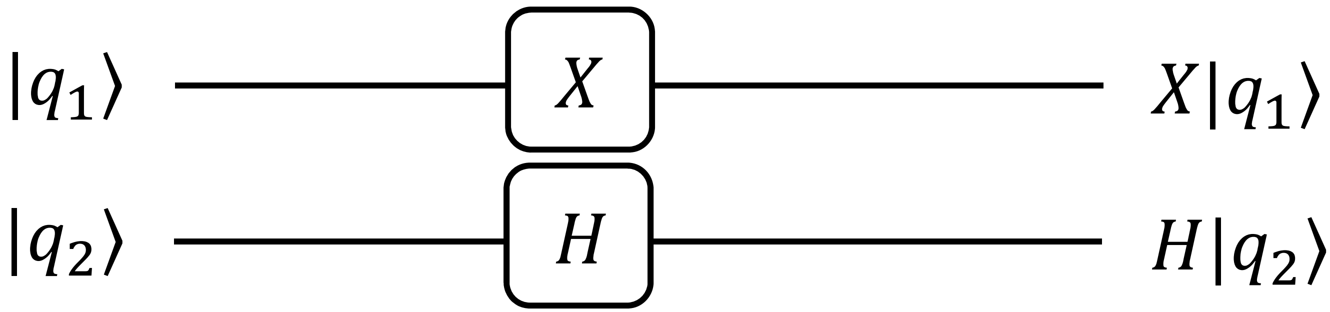
-1 eigenvalue: Probability $|a-b|^2/2$, Outcome $|-\rangle$ — $M_X|\psi\rangle = -1$

Quantum circuits

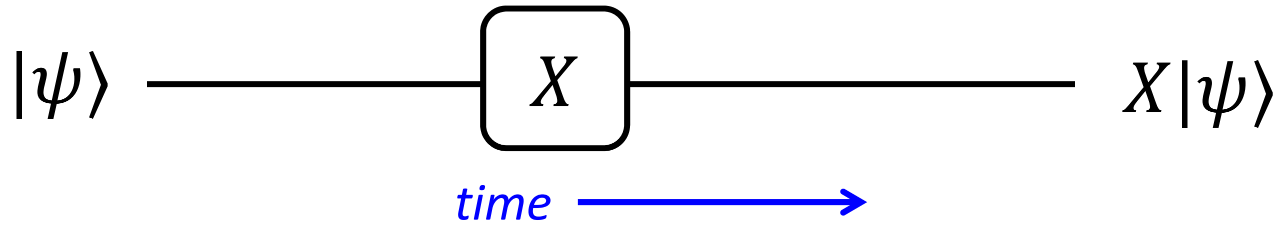


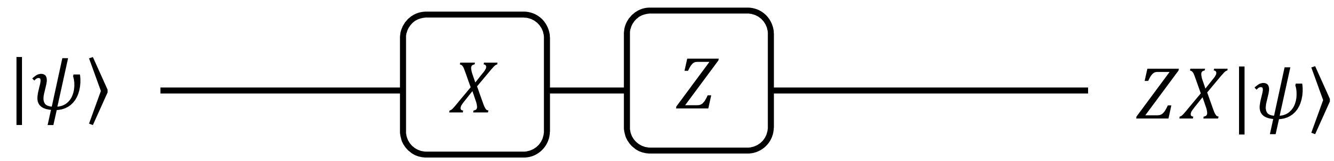


Two qubit circuit

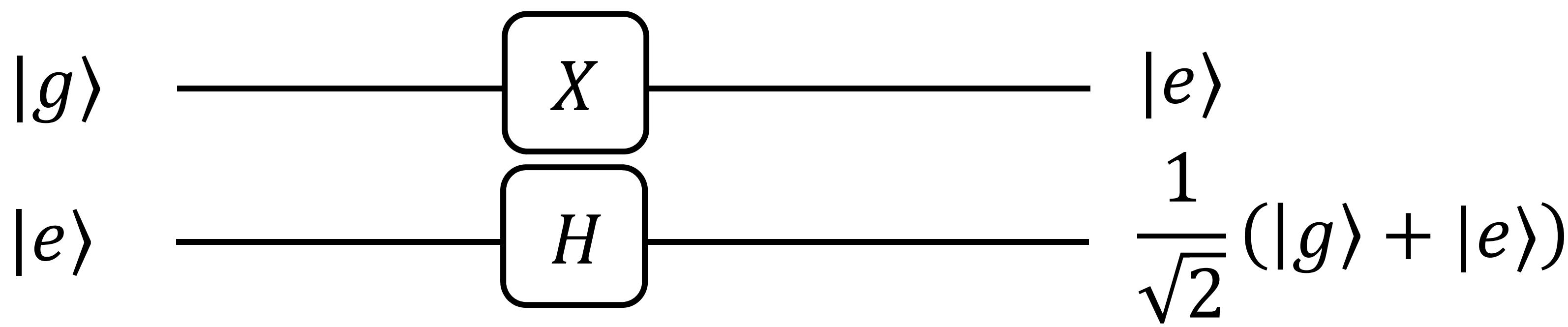


Quantum circuits



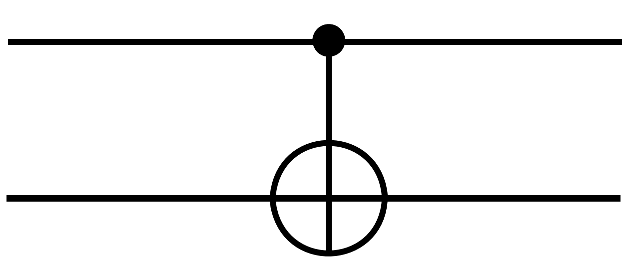


Two qubit circuit



Quantum circuits

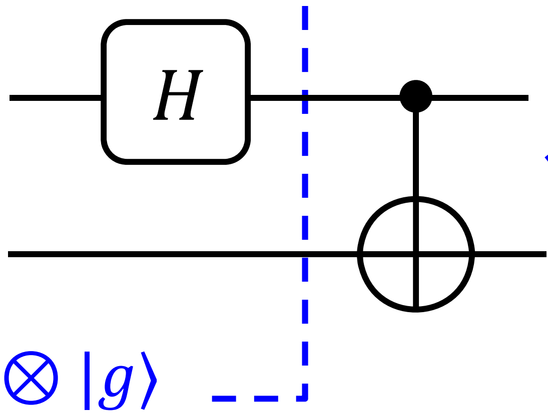
CNOT: $|q_1\rangle$, $|q_2\rangle$



Can result in entangled states

Bell state generation circuit:

$|g\rangle$, $|g\rangle$



$$\frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \otimes |g\rangle$$

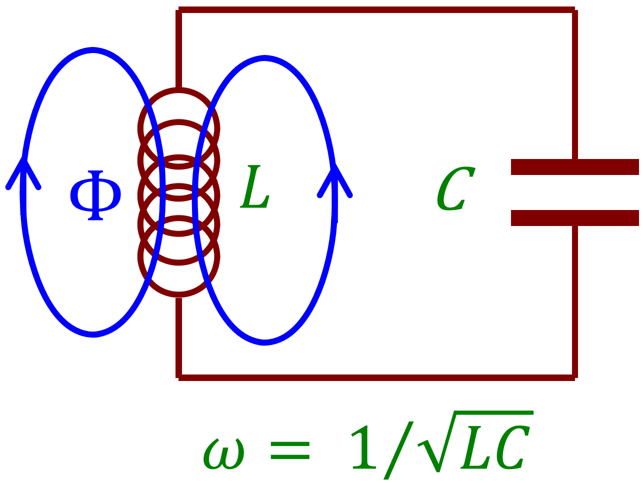
$$= \frac{1}{\sqrt{2}}(|gg\rangle + |eg\rangle)$$

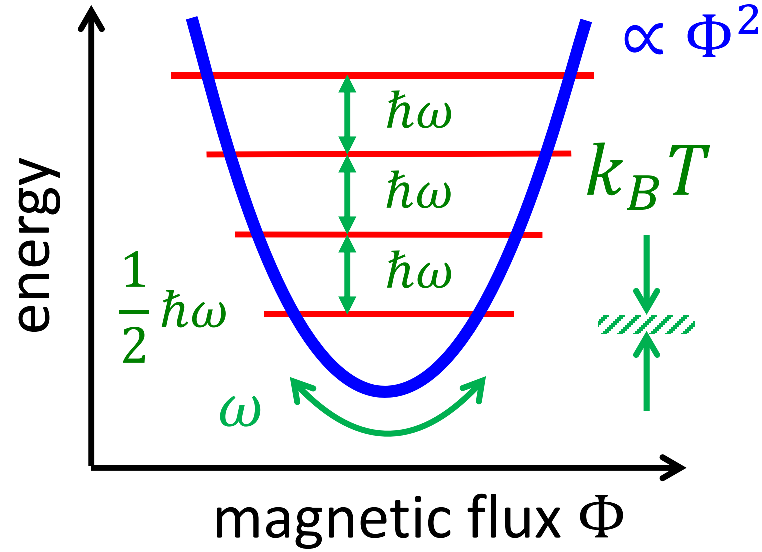
$$\beta_{00} = \frac{1}{\sqrt{2}}(|gg\rangle + |ee\rangle)$$

Inputs determine output:

$$|g\rangle \otimes |g\rangle \Rightarrow \beta_{00}$$
$$|g\rangle \otimes |e\rangle \Rightarrow \beta_{01}$$
$$|e\rangle \otimes |g\rangle \Rightarrow \beta_{10}$$
$$|e\rangle \otimes |e\rangle \Rightarrow \beta_{11}$$

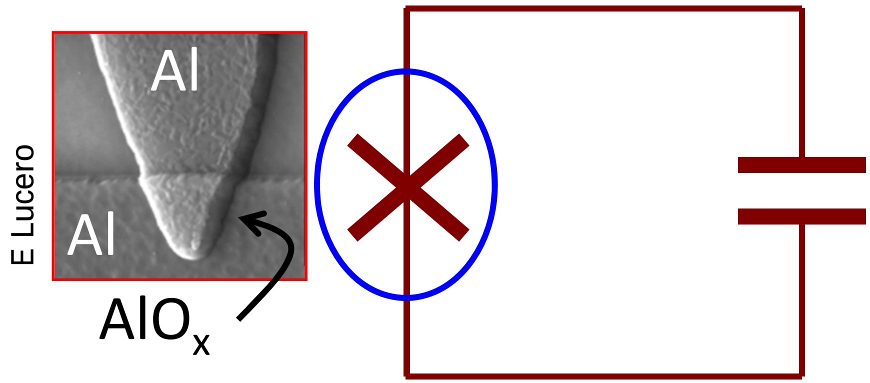
Physical implementation: Transmon qubits

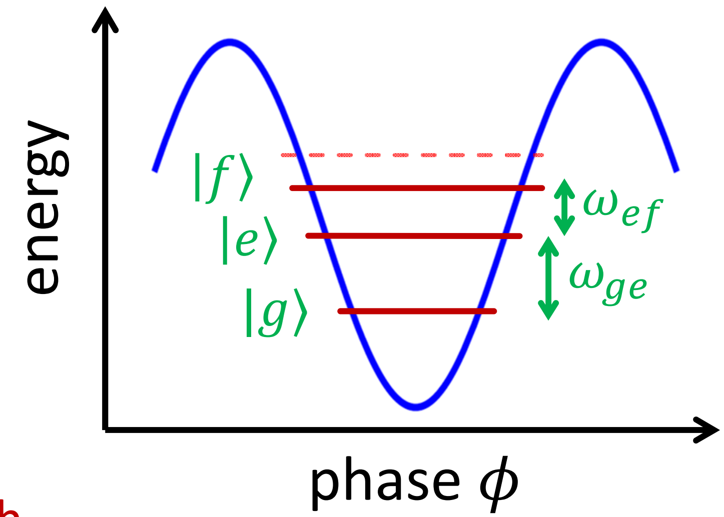




- Microwave frequency circuits: $\omega/2\pi$ ~ few GHz range
- Ground state operation: $k_B T \ll \hbar\omega \Rightarrow T < 100\ \mathrm{mK}$
- Straightforward packaging, cabling, control & measurement electronics
- Need energy level anharmonicity to enable quantum control

Transmon qubits





Josephson junction: A very nonlinear inductor

Inductance L_J changes with each photon in circuit

- Josephson equations:

$$I = I_0 \sin\phi \qquad V = \frac{\hbar}{2e}\frac{d\phi}{dt} = \frac{\hbar}{2e\, I_0 \cos\phi}\frac{dI}{dt}$$

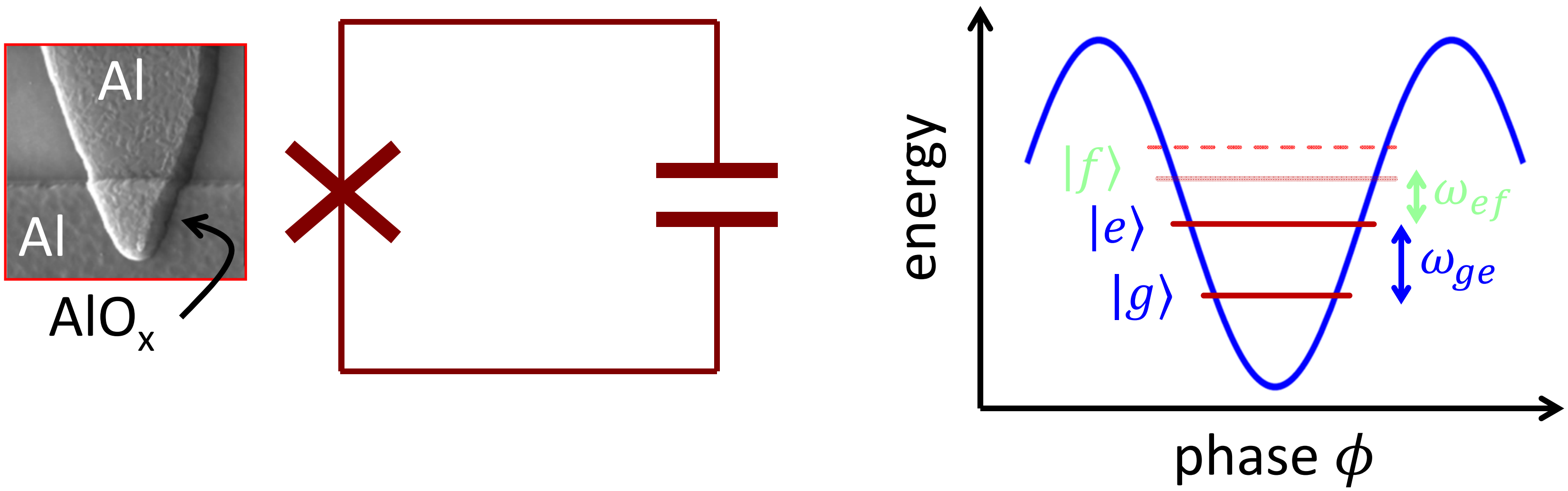
Inductance $L_J(\phi)$

- Circuit Hamiltonian:

$$H = \frac{Q^2}{2C} - E_J \cos\phi \qquad [\phi, Q] = i\hbar$$

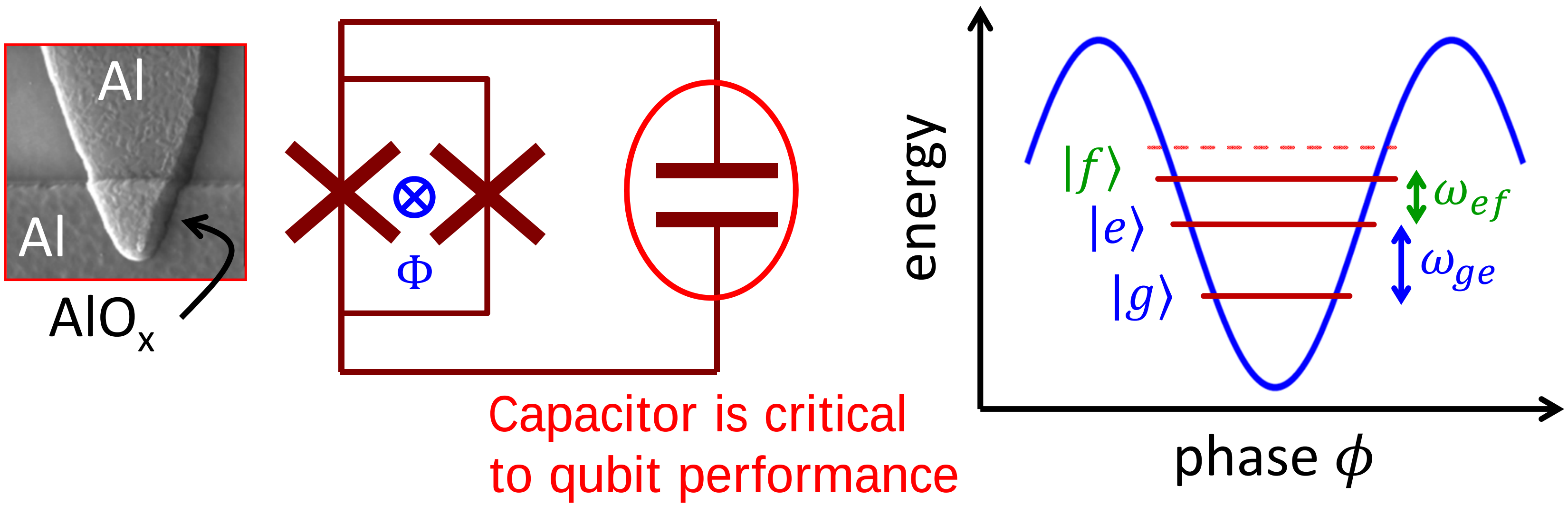
Conjugate variables

Transmon qubits



- Qubit levels $|g\rangle$ and $|e\rangle$
- Qubit frequency $\omega_{ge} \sim 4 - 7\ \mathrm{GHz}$
- Anharmonic at single photon level: $\omega_{ef} \approx 0.95 - 0.97\ \omega_{ge}$

Transmon qubits

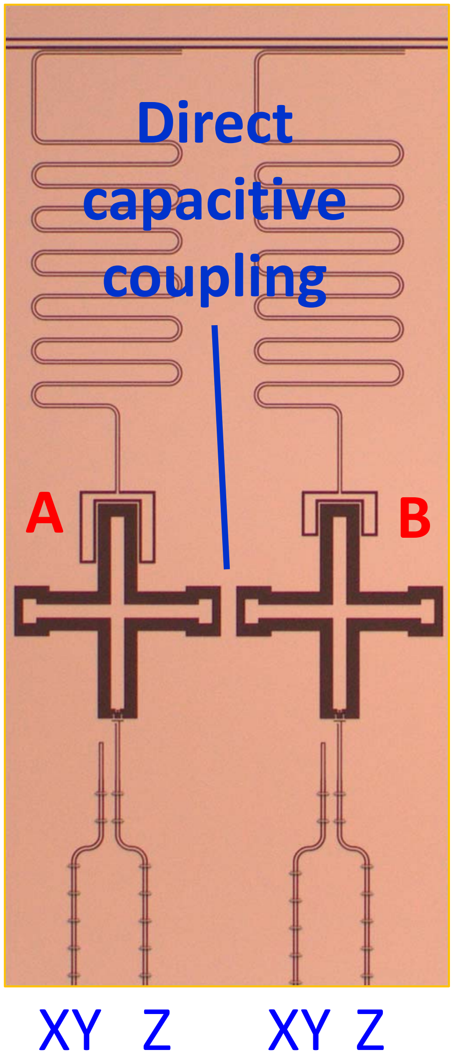


- Qubit levels $|g\rangle$ and $|e\rangle$
- Qubit frequency $\omega_{ge} \sim 4-7\ \mathrm{GHz}$

} X, Y gates: Microwaves at ω_{ge}
Z gate: Tuning ω_{ge}

- Anharmonic at single photon level: $\omega_{ef} \approx 0.95 - 0.97\ \omega_{ge}$
- Can tune ω_{ge} & ω_{ef} by changing magnetic flux through qubit
- Qubit T_1 lifetime probably limited by loss in capacitor
- Qubit T_ϕ probably limited by flux noise

CZ adiabatic gate

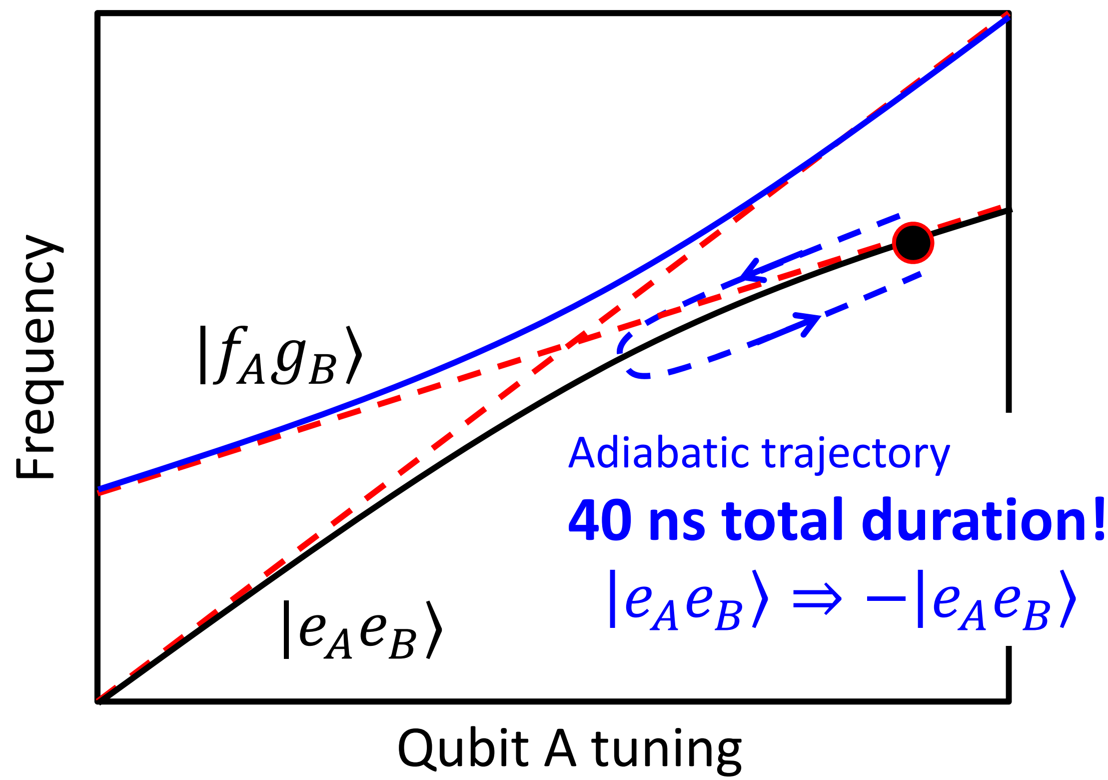


$|e_A e_B\rangle \Rightarrow -|e_A e_B\rangle$

Other states left unchanged

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Avoided crossing $|e_A e_B\rangle - |f_A g_B\rangle$:



Pritzker Nanofabrication Facility

- Recharge-based nanofabrication facility
- In Eckhardt Research Center on UChicago campus (southside Chicago)
- Full professional staff
- Open to all users (academic, industrial…)

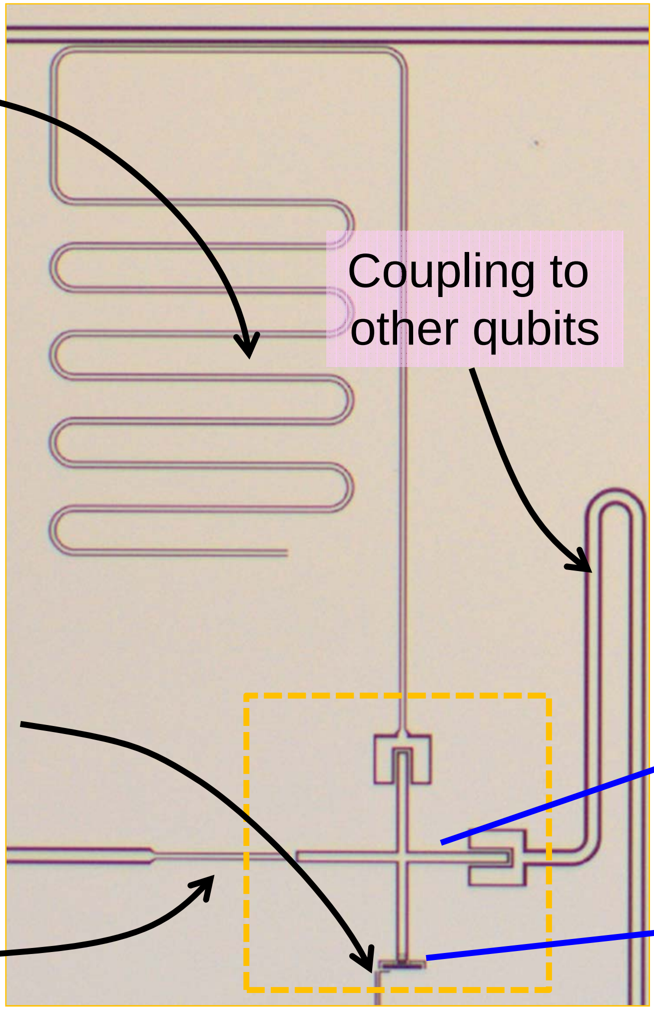
- Sub-10 nm wafer-scale ebeam lithography and 150 mm I-line UV stepper
- Chlorine & fluorine based etching
- Ebeam and sputter deposition of metals & insulators; ALD metal & insulators; wet processing
- SEM, AFM, profilometry, ellipsometry
- Dicing up to 150 mm wafers
- Automatic wire bonder

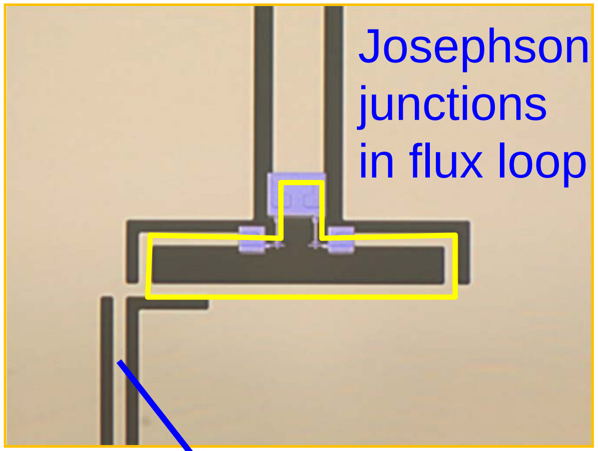
UCSB transmon: "xmon" qubit

Measurement:
- Dispersive resonator readout
- Measure change in phase of few-photon excitation in readout resonator
- Projective & accurate

Z rotations:
- Flux tuning varies L
- Changes qubit frequency

X and Y rotations:
- Microwaves at qubit frequency





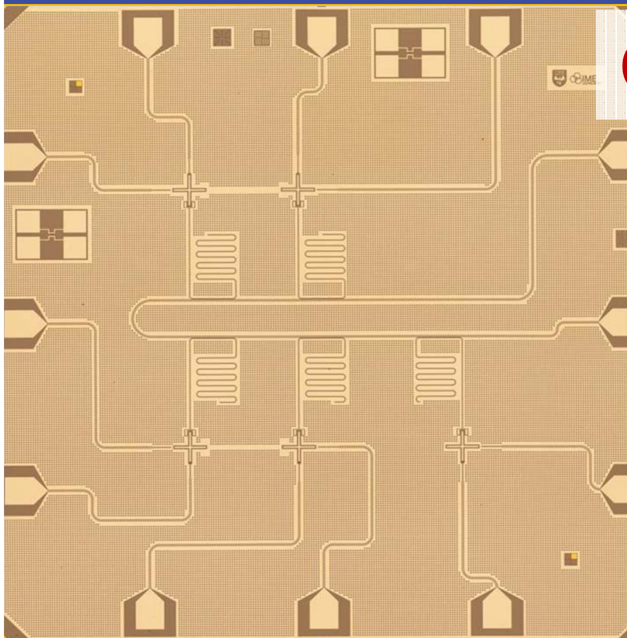
Z control

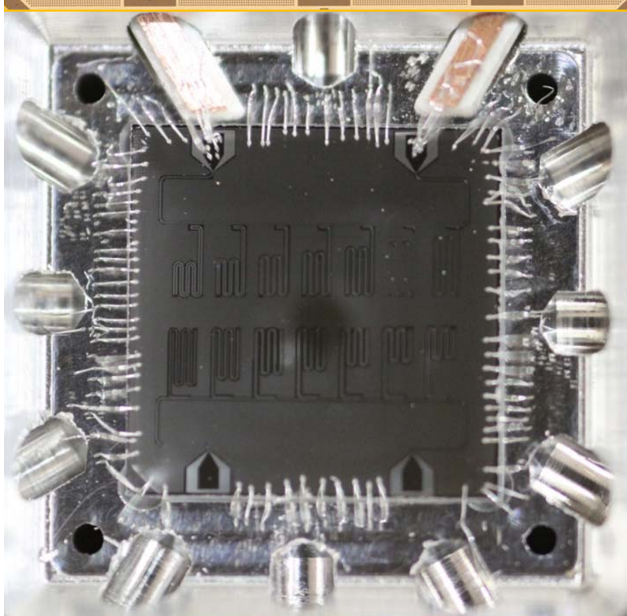
Capacitance C

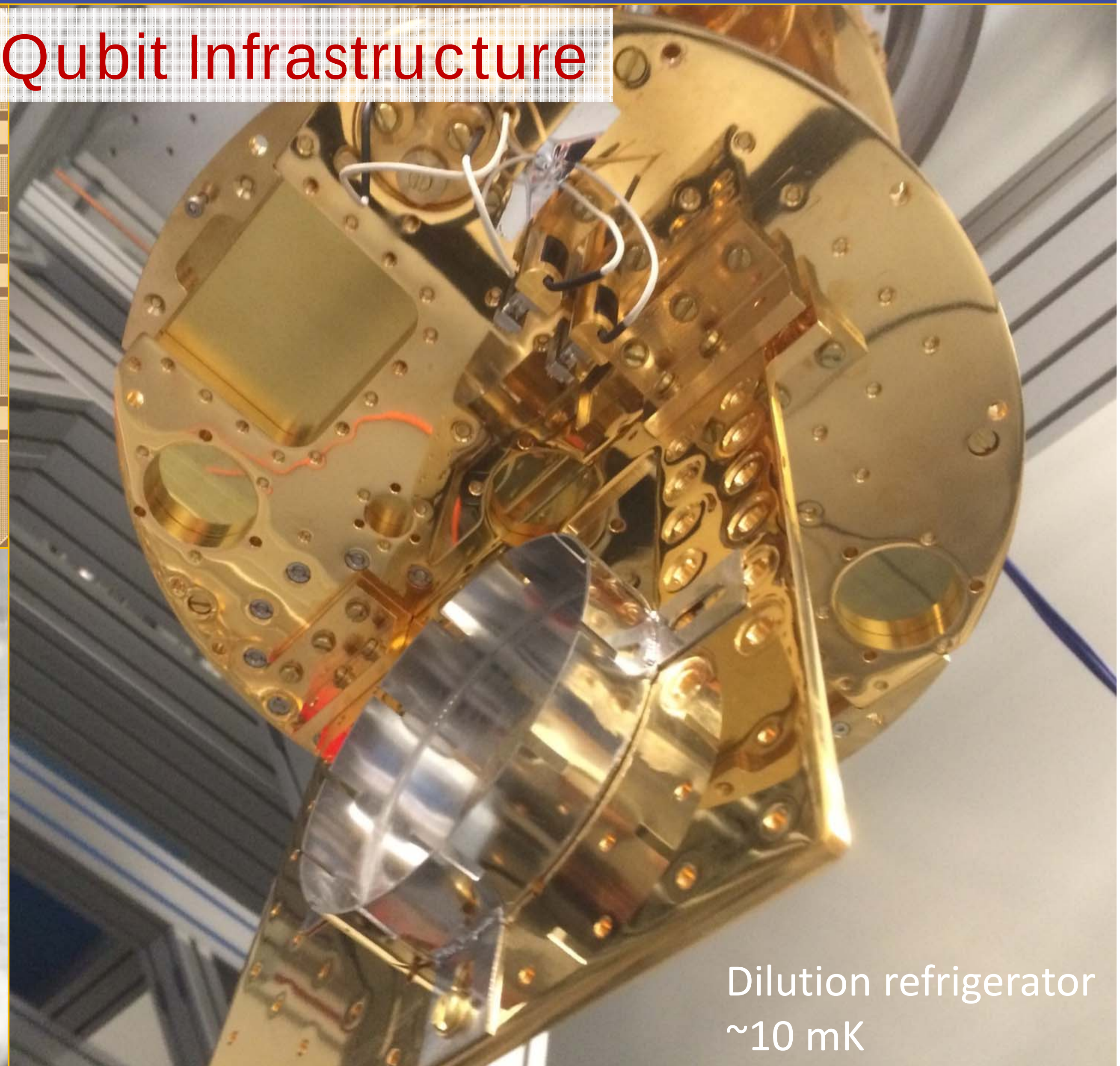
Josephson junctions (tunable inductance L)

Barends et al. (PRL, 2013)

Qubit Infrastructure

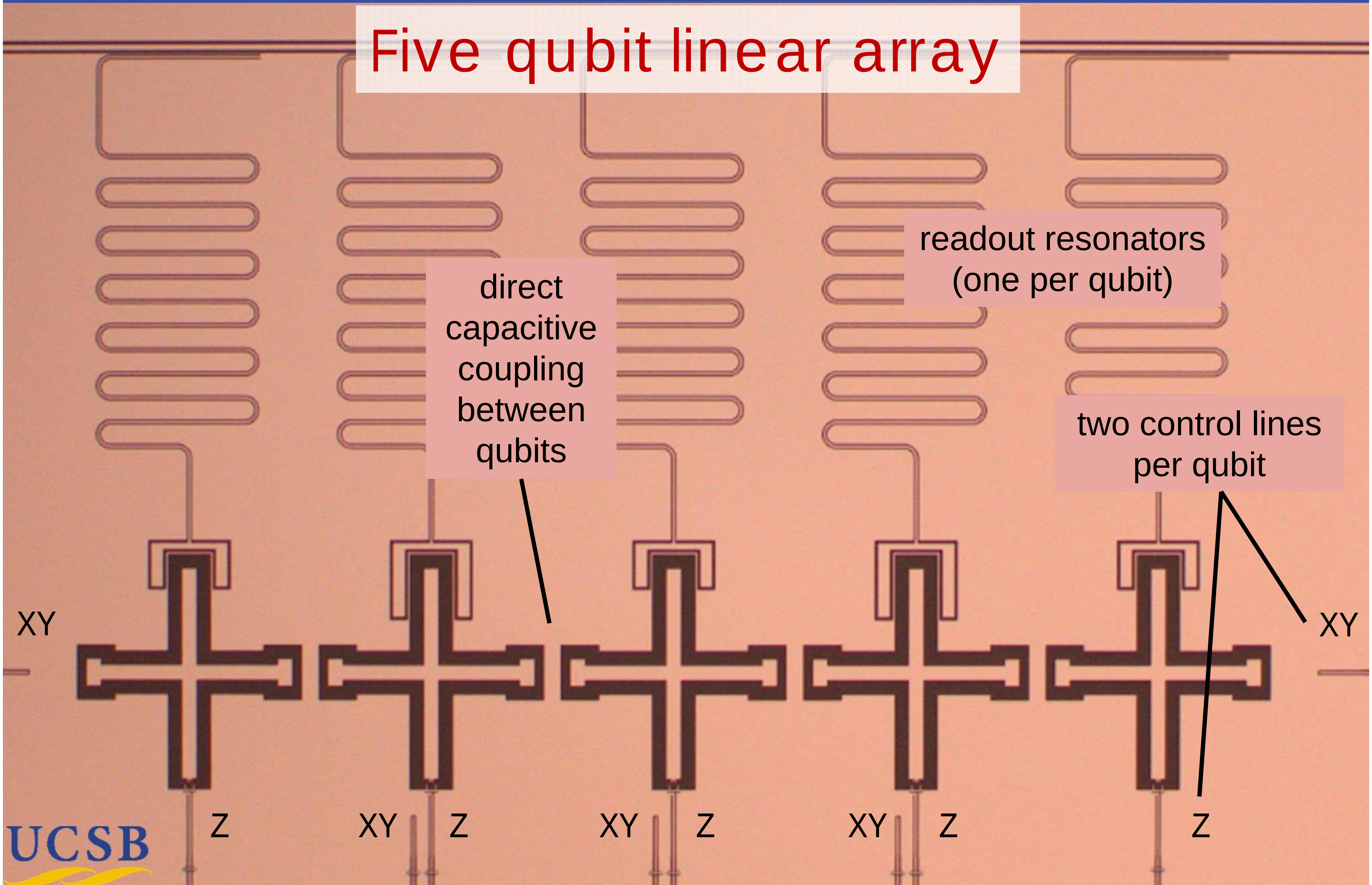






Dilution refrigerator
~10 mK

Five qubit linear array



Barends et al. (Nature, 2014)

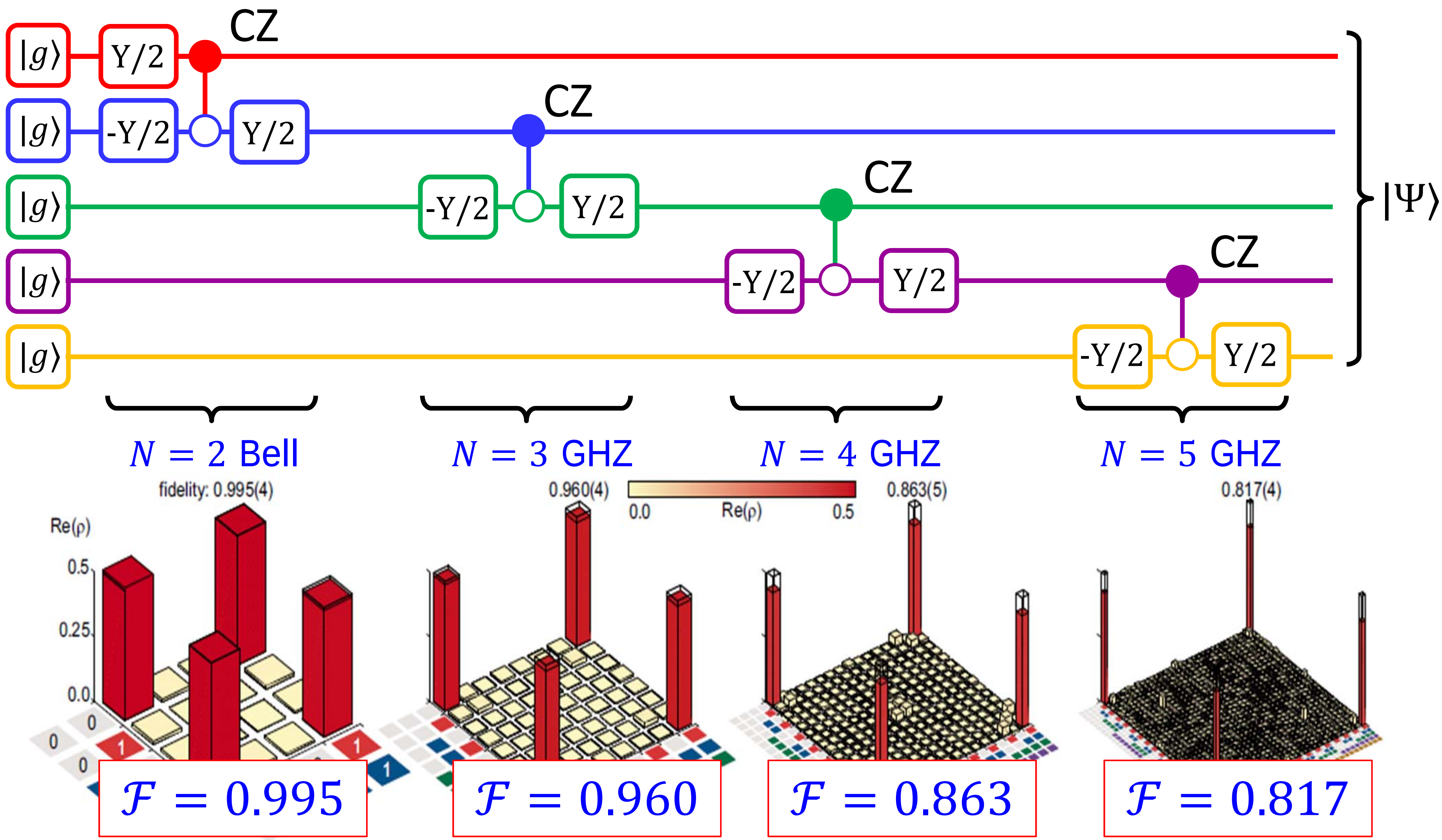
Experimental status (UCSB)

- Complete set of single qubit gates needed to execute an arbitrary quantum algorithm
- All single qubit Clifford gates operate with fidelity > 99.9% (randomized benchmarking)
- Controlled Z gate (equivalent to CNOT):
  - 40 ns execution time
  - 99.5% fidelity
- State preparation and measurement fidelity ~ 90%

Gate	Fidelity ($\pm$0.03)
X	99.92
Y	99.92
X/2	99.93
Y/2	99.95
-X	99.92
-Y	99.91
-X/2	99.93
-Y/2	99.95
H	99.91
I	99.95
S (Z/2)	99.92
CZ	99.5

Barends et al. (Nature, 2014)

Programming a 5 qubit GHZ state

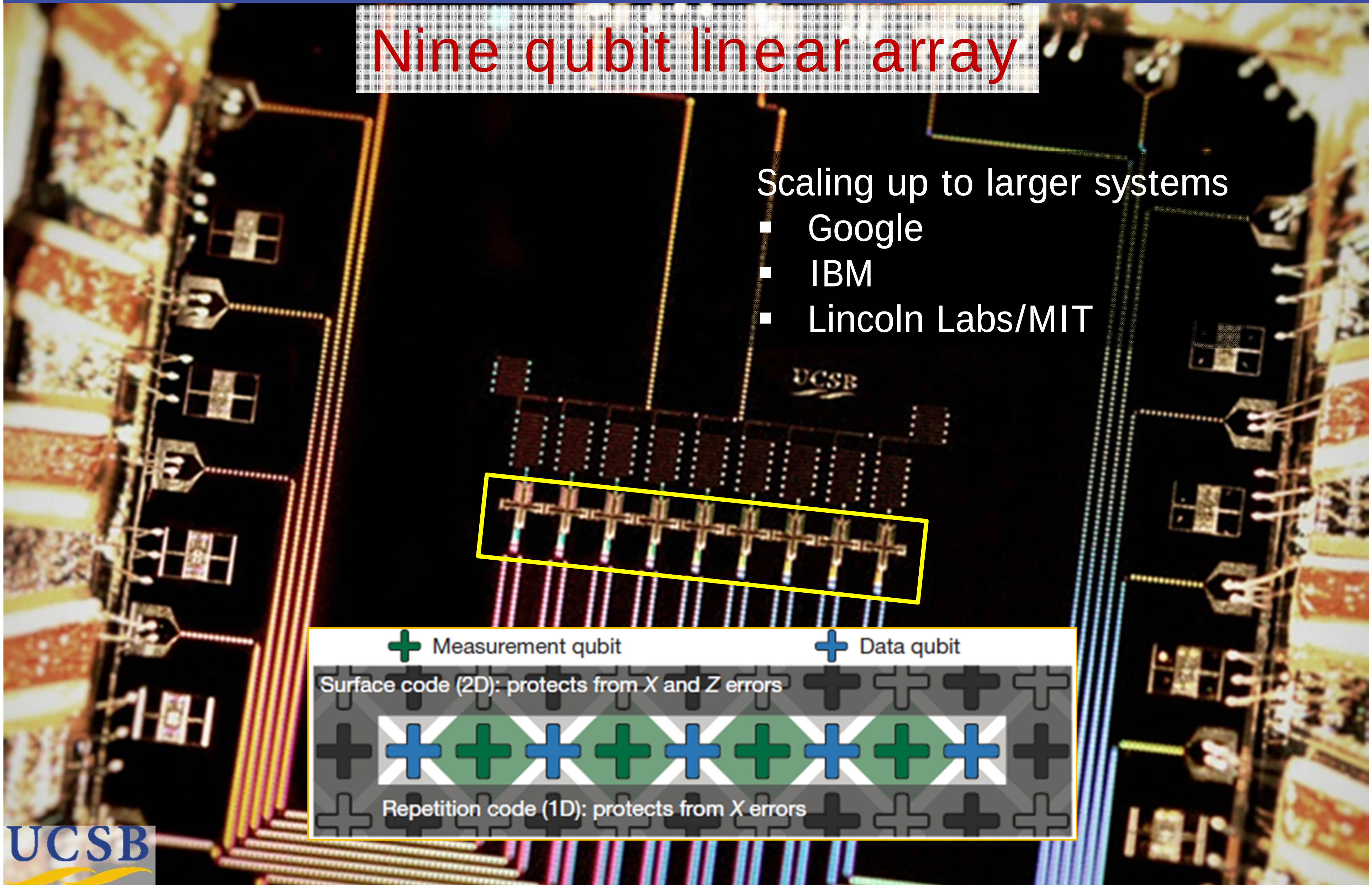


Barends et al. (Nature, 2014)

Nine qubit linear array

Scaling up to larger systems

- Google
- IBM
- Lincoln Labs/MIT



Kelly et al. (Nature, 2015)

Error Models

One accepted and relatively straightforward way to introduce error-producing events in quantum computation is to use Kraus operators. We will not cover that formalism here.

Consider the following less formal approach:

$$|\mathcal{E}\rangle|g\rangle \Rightarrow a_1|\mathcal{E}_1\rangle|g\rangle + a_2|\mathcal{E}_2\rangle|e\rangle$$

environment

$|a_1|$ slightly less than 1

$1-\sqrt{p_e}$

$|a_2|$ small, $\sqrt{p_e}$

$$|\mathcal{E}\rangle|e\rangle \Rightarrow a_3|\mathcal{E}_3\rangle|g\rangle + a_4|\mathcal{E}_4\rangle|e\rangle$$

$|a_3|$ small, $\sqrt{p_g}$

$|a_4|$ slightly less than 1

$1-\sqrt{p_g}$

Error Models

Projectors onto $|g\rangle$ and $|e\rangle$:

$$P_g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(I + Z) \qquad P_e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(I - Z)$$

$$P_g + P_e = I$$

We can write:
$$|\mathcal{E}\rangle|g\rangle \Rightarrow a_1|\mathcal{E}_1\rangle|g\rangle + a_2|\mathcal{E}_2\rangle|e\rangle$$
$$= (a_1|\mathcal{E}_1\rangle X + a_2|\mathcal{E}_2\rangle I)|e\rangle$$
$$|\mathcal{E}\rangle|e\rangle \Rightarrow a_3|\mathcal{E}_3\rangle|g\rangle + a_4|\mathcal{E}_4\rangle|e\rangle$$
$$= (a_3|\mathcal{E}_3\rangle I + a_4|\mathcal{E}_4\rangle X)|g\rangle$$

Then with $|\psi\rangle = \alpha|g\rangle + \beta|e\rangle$ we can write

$$|\mathcal{E}\rangle|\psi\rangle \Rightarrow (a_1|\mathcal{E}_1\rangle X + a_2|\mathcal{E}_2\rangle I)P_g|\psi\rangle +$$
$$+(a_3|\mathcal{E}_3\rangle I + a_4|\mathcal{E}_4\rangle X)P_e|\psi\rangle$$

Error Models

Copying from previous slide, we can then write

$$|\mathcal{E}\rangle|\psi\rangle \Rightarrow (a_1|\mathcal{E}_1\rangle X + a_2|\mathcal{E}_2\rangle I)P_g|\psi\rangle + (a_3|\mathcal{E}_3\rangle I + a_4|\mathcal{E}_4\rangle X)P_e|\psi\rangle$$

$$= \left\{(a_1|\mathcal{E}_1\rangle X + a_2|\mathcal{E}_2\rangle I)\frac{I+Z}{2} + (a_3|\mathcal{E}_3\rangle I + a_4|\mathcal{E}_4\rangle X)\frac{I-Z}{2}\right\}|\psi\rangle$$

$$= \left\{\frac{a_2|\mathcal{E}_2\rangle + a_3|\mathcal{E}_3\rangle}{2} I + \frac{a_1|\mathcal{E}_1\rangle + a_4|\mathcal{E}_4\rangle}{2} X + \frac{a_1|\mathcal{E}_1\rangle - a_4|\mathcal{E}_4\rangle}{2} XZ + \frac{a_2|\mathcal{E}_2\rangle - a_4|\mathcal{E}_4\rangle}{2} Z\right\}|\psi\rangle$$

$XZ = -iY$

$$= \left\{\sqrt{1-p}\,|\mathcal{E}_I\rangle I + \sqrt{p_X}\,|\mathcal{E}_X\rangle X + \sqrt{p_Y}\,|\mathcal{E}_Y\rangle Y + \sqrt{p_Z}\,|\mathcal{E}_Z\rangle Z\right\}|\psi\rangle$$

We can treat the environment interaction as generating discrete X, Y or Z errors!

Summary of Monday's lecture

- Superconducting qubits have single qubit fideli[illegible]5% (for gate times ~50 ns)
- Qubit idle ~ same fidelity – due to inter[illegible]th environment
- To do useful computing need fideliti[illegible]der

$$99.999999[illegible]99999\%$$

- How to get there?
- Build perfection fro[illegible]ction – stabilizers & surface codes to create l[illegible]bits
- Models pred[illegible]al qubits with 99.99% fidelity can form logical q[illegible] very high levels of fidelity, assuming:
  - [illegible]trong long-range correlated errors
  - Large circuits can be built maintaining fidelities
  - Large entangled states don't need any new physics
  - ...

CNOT fidelity 99.5%: Significant challenge!

Slides for Capri School: Surface Codes

Andrew N Cleland

Capri April 24-28 2017

Lecture 2: The Surface Code

- Codewords & error syndromes
- Surface code architecture
- Stabilizer operations
- Quiescent state
- Single qubit errors
- Logical qubits, logical operators

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Summary of Monday's lecture

- Superconducting qubits have single qubit fidelit[illegible]5% (for gate times ~50 ns)
- Qubit idle ~ same fidelity – due to inter[illegible]th environment
- To do useful computing need fideliti[illegible]der

99.999999[illegible]99999%

- How to get there?
- Build perfection fro[illegible]ction – stabilizers & surface codes to create [illegible]bits
- Models predi[illegible]al qubits with 99.99% fidelity can form logical q[illegible] very high levels of fidelity, assuming:
  - [illegible]trong long-range correlated errors
  - [illegible]arge circuits can be built maintaining fidelities
  - Large entangled states don't need any new physics
  - ...

CNOT fidelity 99.5%: Significant challenge!

Literature

- S. B. Bravyi and A. Y. Kitaev, arXiv:quant-ph/9811052
- E. Dennis, A. Y. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. **43**, 4452 (2002)
- A. Y. Kitaev, in *Proc. 3rd Intl. Conf. Quant. Comm. Comp. Meas.,* ed. O. Hirota, A. S. Holevo, C. M. Caves (Plenum Press, New York, 1997)
- A. Y. Kitaev, Russian Math Surveys **52**, 1191 (1997)
- A. Y. Kitaev, Ann. Phys. **303**, 2 (2003)
- M. H. Freedman and D. A. Meyer, Found. Comp. Math. **1**, 325 (2001)
- C. Wang, J. Harrington, and J. Preskill, Ann. Phys. **303**, 31 (2003)
- R. Raussendorf, J. Harrington, and K. Goyal, Ann. Phys. **321**, 2242 (2006)
- R. Raussendorf and J. Harrington, Phys. Rev. Lett. **98**, 190504 (2007)
- R. Raussendorf, J. Harrington, and K. Goyal, New J. Phys. **9**, 199 (2007)
- A.G. Fowler, A.M. Stephens, P. Groszkowski, Phys. Rev. A **80**, 052312 (2009)
- A.G. Fowler, D.S. Wang, L.C.L. Hollenberg, Quant. Inf. Comp. **11**, 8 (2011)
- D.S. Wang, A.G. Fowler, L.C.L. Hollenberg, Phys. Rev. A **83**, 020302 (2011)
- A.G. Fowler, A.C. Whiteside, L.C.L. Hollenberg, Phys. Rev. Lett. **108**, 180501 (2012)
- A.G. Fowler, M. Mariantoni, J.M. Martinis, A.N. Cleland, Phys Rev A **86**, 032324 (2012)

Error Syndromes & Codewords: Bit-flips

Simple example: Bit-flip detection **codewords**

$$|g_L\rangle = |ggg\rangle \qquad |e_L\rangle = |eee\rangle$$

Simple quantum circuit enables encoding

$$|\psi\rangle = a|g\rangle + b|e\rangle \;\Rightarrow a|ggg\rangle + b|eee\rangle$$

$$|\psi_L\rangle = a|g_L\rangle + b|e_L\rangle$$

Qubits 1 & 3 can out-vote qubit 2

Error example: X bit-flip error on qubit 2 (X_2 error):

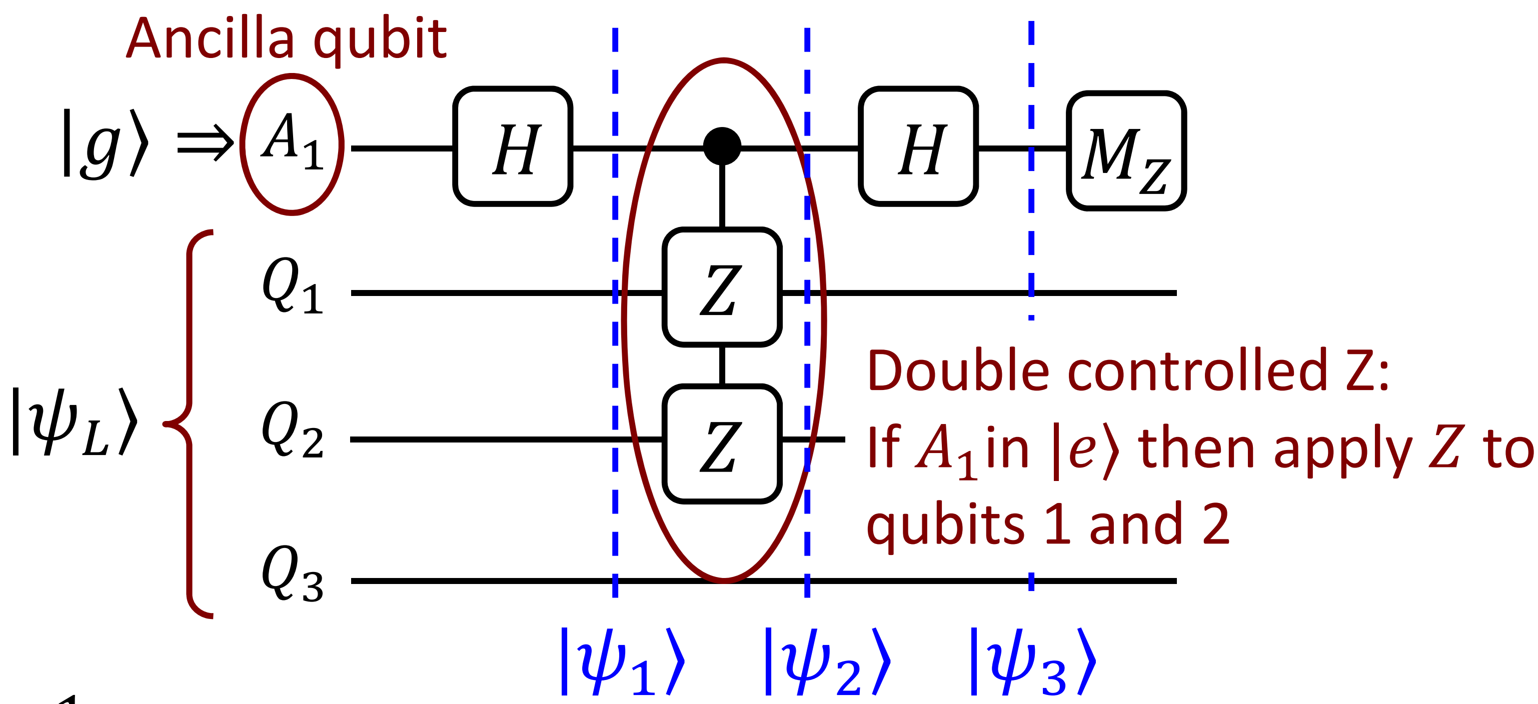
$$X_2(a|ggg\rangle + b|eee\rangle) = a|geg\rangle + b|ege\rangle$$

How do we detect & fix this error (or a single X_1 or X_3 error)?

We build an **error syndrome** circuit

Error Syndromes & Error Codes

$$X_2(a|ggg\rangle + b|eee\rangle) = a|geg\rangle + b|ege\rangle$$



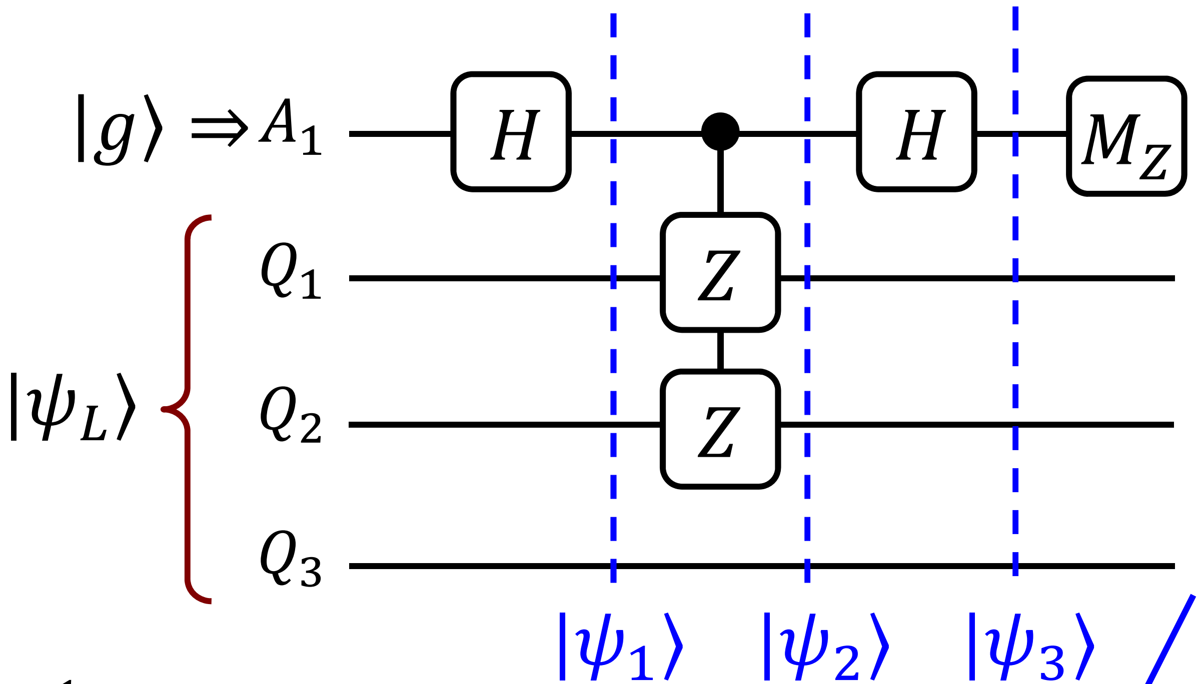
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \otimes |\psi_L\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle Z_1 Z_2)|\psi_L\rangle$$

$$|\psi_3\rangle = \frac{1}{2}(|g\rangle + |e\rangle + |g\rangle Z_1 Z_2 - |e\rangle Z_1 Z_2)|\psi_L\rangle$$

Error Syndromes & Error Codes

$$X_2(a|ggg\rangle + b|eee\rangle) = a|geg\rangle + b|ege\rangle$$



Measure the ancilla along Z:

If $M_Z = +1$: Result is $|g\rangle|\psi_L\rangle$ and $Z_1Z_2 = 1$

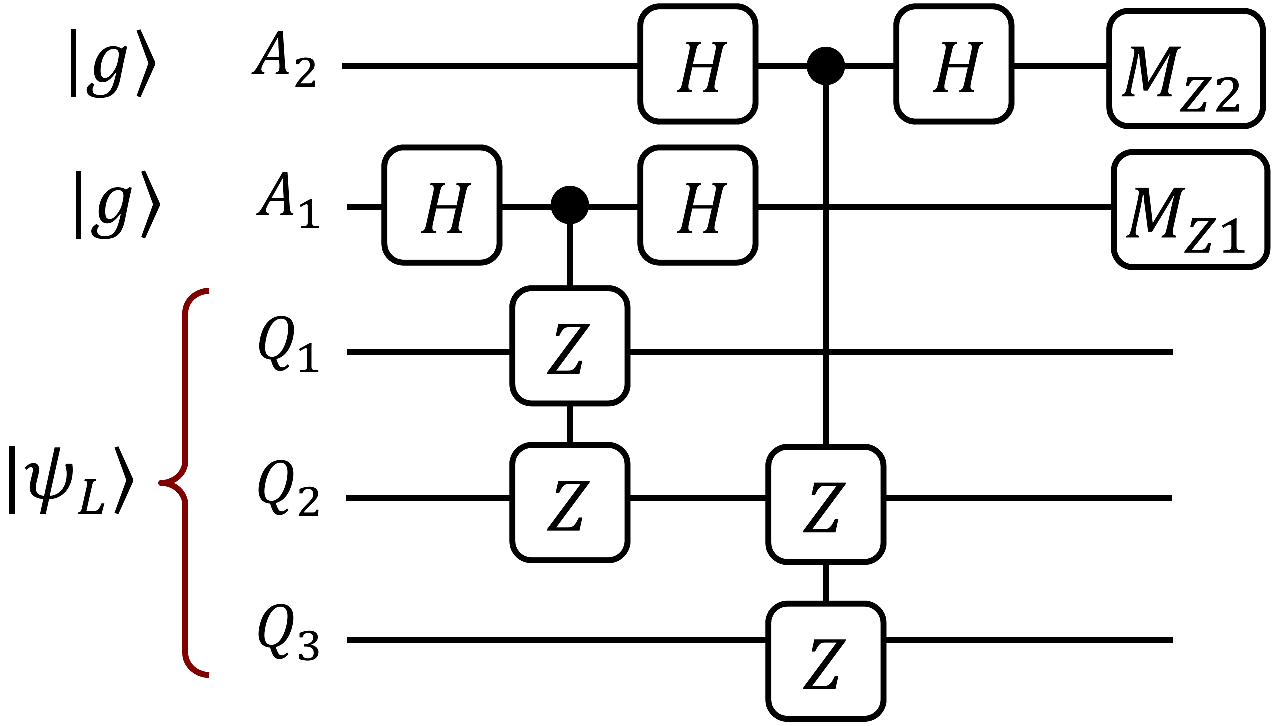
If $M_Z = -1$: Result is $|e\rangle|\psi_L\rangle$ and $Z_1Z_2 = -1$

Error syndrome

$$|\psi_3\rangle = \frac{1}{2}(|g\rangle + |e\rangle + |g\rangle Z_1Z_2 - |e\rangle Z_1Z_2)|\psi_L\rangle$$

$$= |g\rangle\frac{I + Z_1Z_2}{2}|\psi_L\rangle + |e\rangle\frac{I -}{\ }$$

Error Syndromes & Error Codes



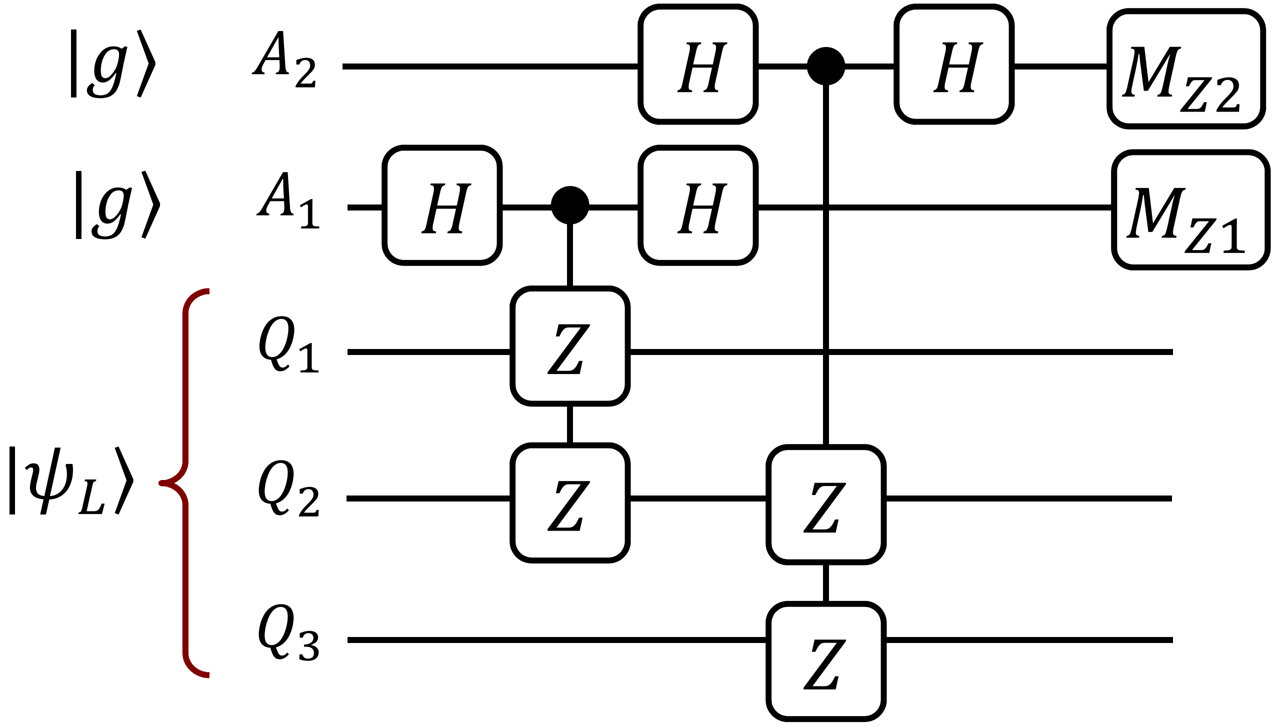
No error: No syndrome, $M_{Z1} = M_{Z2} = +1$

X_1 error: A_1 signals error with $M_{Z1} = -1$

X_2 error: A_1 and A_2 signal errors with $M_{Z1} = M_{Z2} = -1$

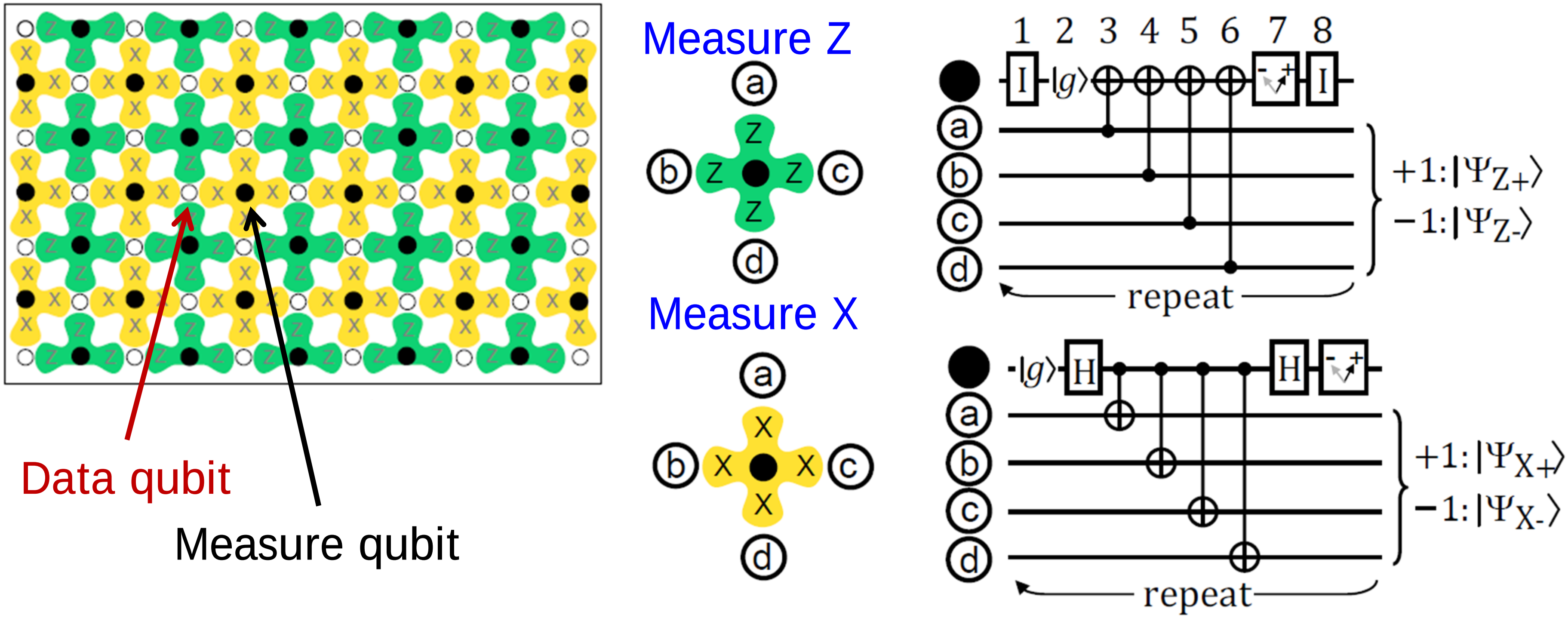
X_3 error: A_2 signals errors with $M_{Z2} = -1$

Error Syndromes & Error Codes



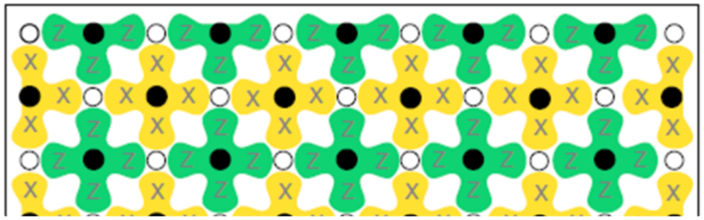
- This 3 qubit codeword+2 ancilla error syndrome circuit can only detect & identify single X bit-flips
- There are (famous) 5 qubit codeword+4 ancilla and 7 qubit codeword+6 ancilla syndromes that can detect and identify single X, Y and Z errors on any codeword qubit

Surface code cycle & eigenstates



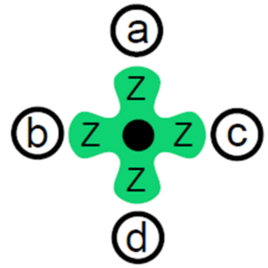
- Data qubits store computational state
- Measure qubit stabilize data qubits
- Measure Z stabilizes $ZZZZ$ product of data qubits
- Measure X stabilizes $XXXX$ product of data qubits

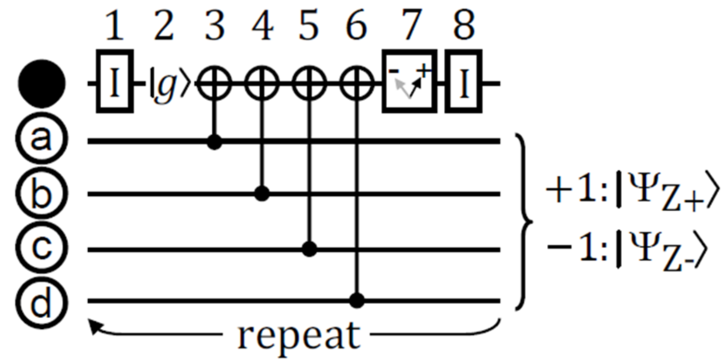
Surface code cycle & eigenstates



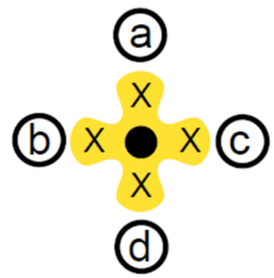
Eigenvalue	$\hat{Z}_a\hat{Z}_b\hat{Z}_c\hat{Z}_d$	$\hat{X}_a\hat{X}_b\hat{X}_c\hat{X}_d$
+1	$\|gggg\rangle$	$\|++++\rangle$
	$\|ggee\rangle$	$\|++--\rangle$
	$\|geeg\rangle$	$\|+--+\rangle$
	$\|eegg\rangle$	$\|--++\rangle$
	$\|egge\rangle$	$\|-++-\rangle$
	$\|gege\rangle$	$\|+-+-\rangle$
	$\|egeg\rangle$	$\|-+-+\rangle$
	$\|eeee\rangle$	$\|----\rangle$
−1	$\|ggge\rangle$	$\|+++-\rangle$
	$\|ggeg\rangle$	$\|++-+\rangle$
	$\|gegg\rangle$	$\|+-++\rangle$
	$\|eggg\rangle$	$\|-+++\rangle$
	$\|geee\rangle$	$\|+---\rangle$
	$\|egee\rangle$	$\|-+--\rangle$
	$\|eege\rangle$	$\|--+-\rangle$
	$\|eeeg\rangle$	$\|---+\rangle$

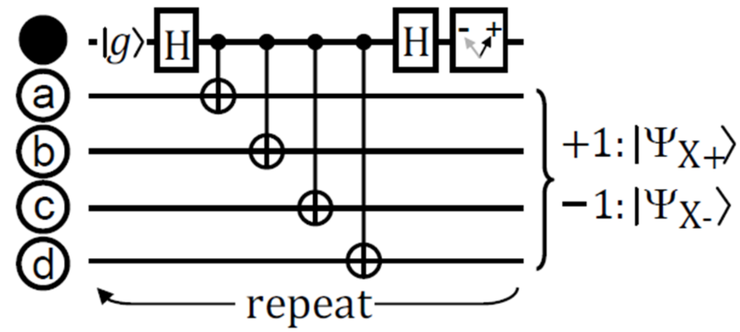
Measure Z





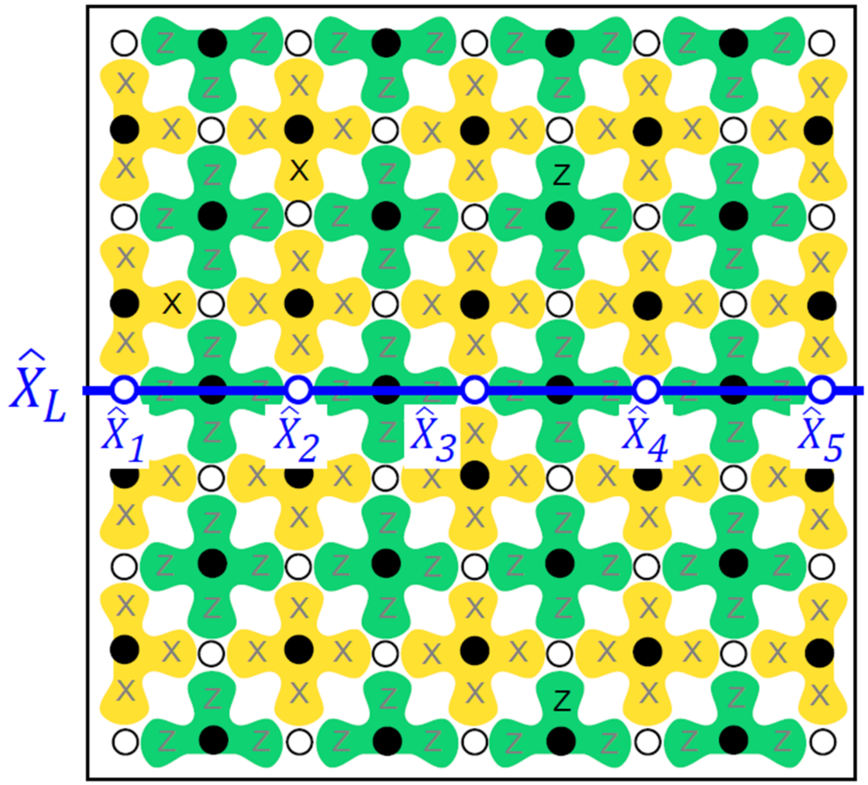
Measure X



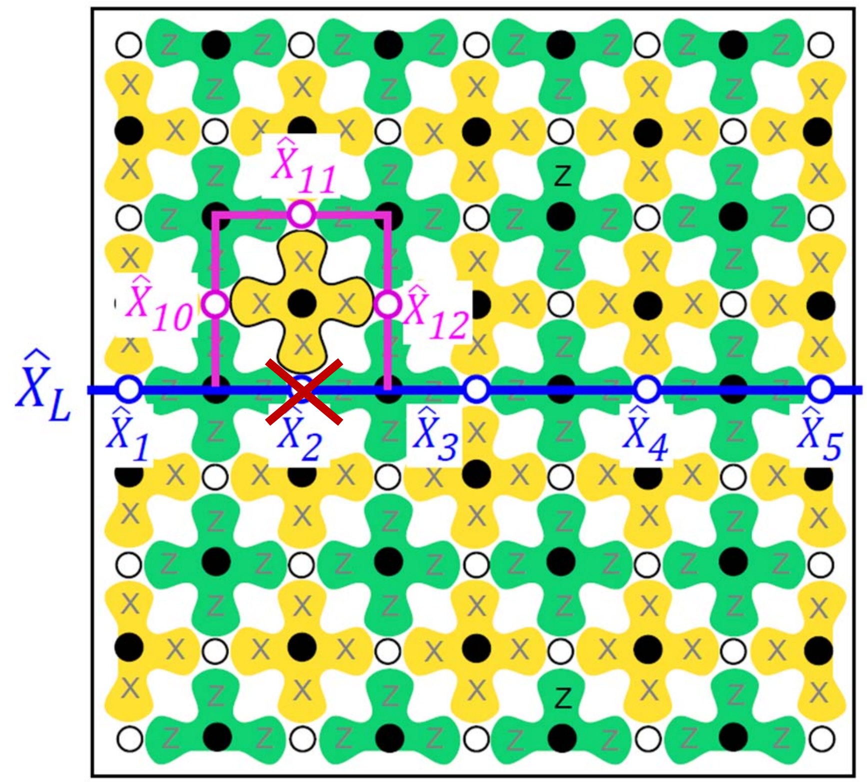


- Data qubit states stabilized by measure qubits
- Single bit-flip (phase-flip) detected by change in Z (X) measurement outcome
- Stabilizer errors don't repeat in time

Surface code logical operators



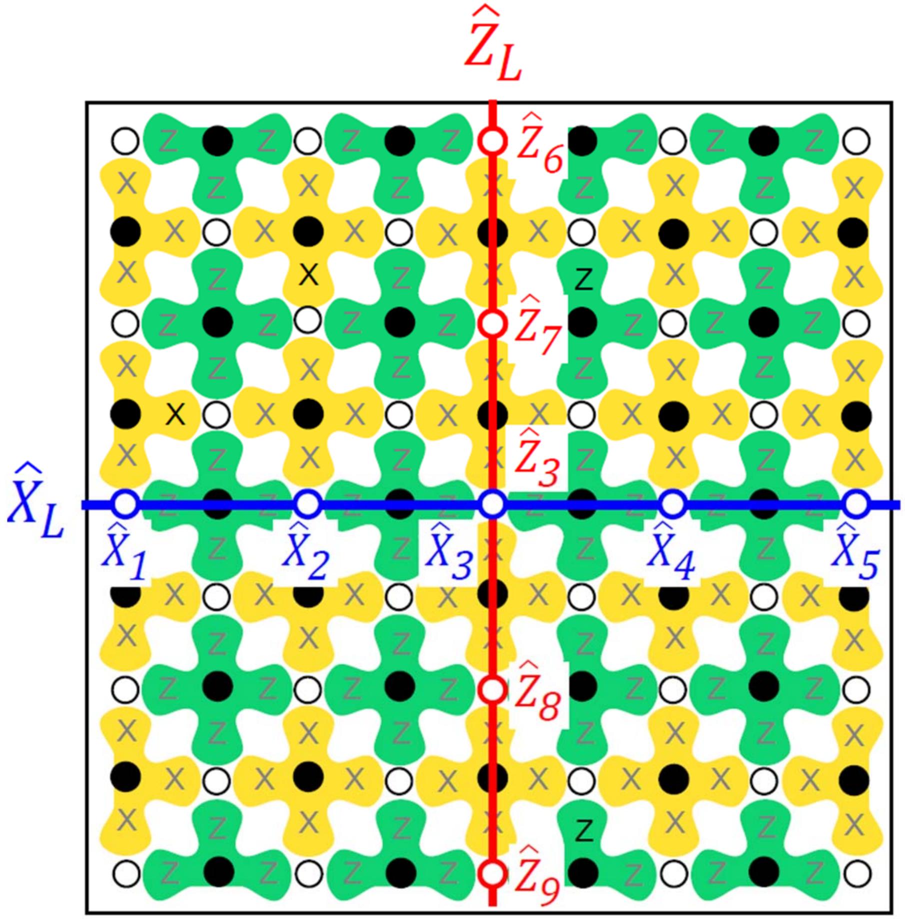
$$X_L = X_1 X_2 X_3 X_4 X_5$$



$$X_L \times \underbrace{X_{10} X_{11} X_{12} X_2}_{} \Leftrightarrow X_L'$$

Stabilizer with value ± 1

Surface code logical operators



Array of data & measure qubits forms a logical qubit (81 physical qubits)

Logical X: $X_L = X_1X_2X_3X_4X_5$

Logical Z: $Z_L = Z_6Z_7Z_3Z_8Z_9$

Same properties as X_L

$$X_LZ_L = X_1X_2X_3X_4X_5 \times Z_6Z_7Z_3Z_8Z_9$$
$$= -Z_6Z_7Z_3Z_8Z_9 \times X_1X_2X_3X_4X_5$$
$$= -Z_LX_L$$

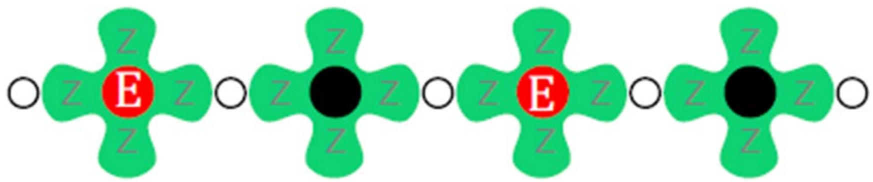
- X_L and Z_L anti-commute, as desired

$$X_L^2 = Z_L^2 = I \qquad Y_L = iX_LZ_L$$

- All required properties

Errors & misidentification

- Measure Z qubits 1 and 3 change measurement sign
- Error!



- Could be due to errors on data qubits 2 and 3

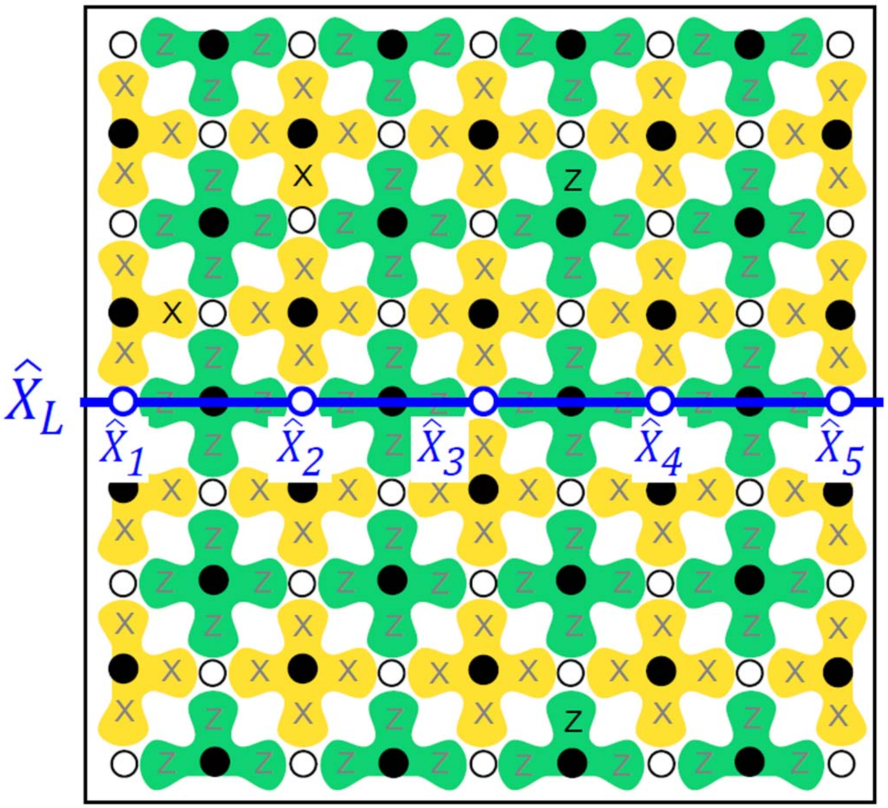
- Could be due to errors on data qubits 1 and 4 and 5

  - Probability of any single qubit error is p (assumed small!)
  - Probability of two qubit error is p^2 — This is more likely!
  - Probability of three qubit error is $p^3 \ll p^2$

Natural conclusion is two-qubit error

If actually a 3-qubit error, this assumption generates a logical error!

Errors & misidentification

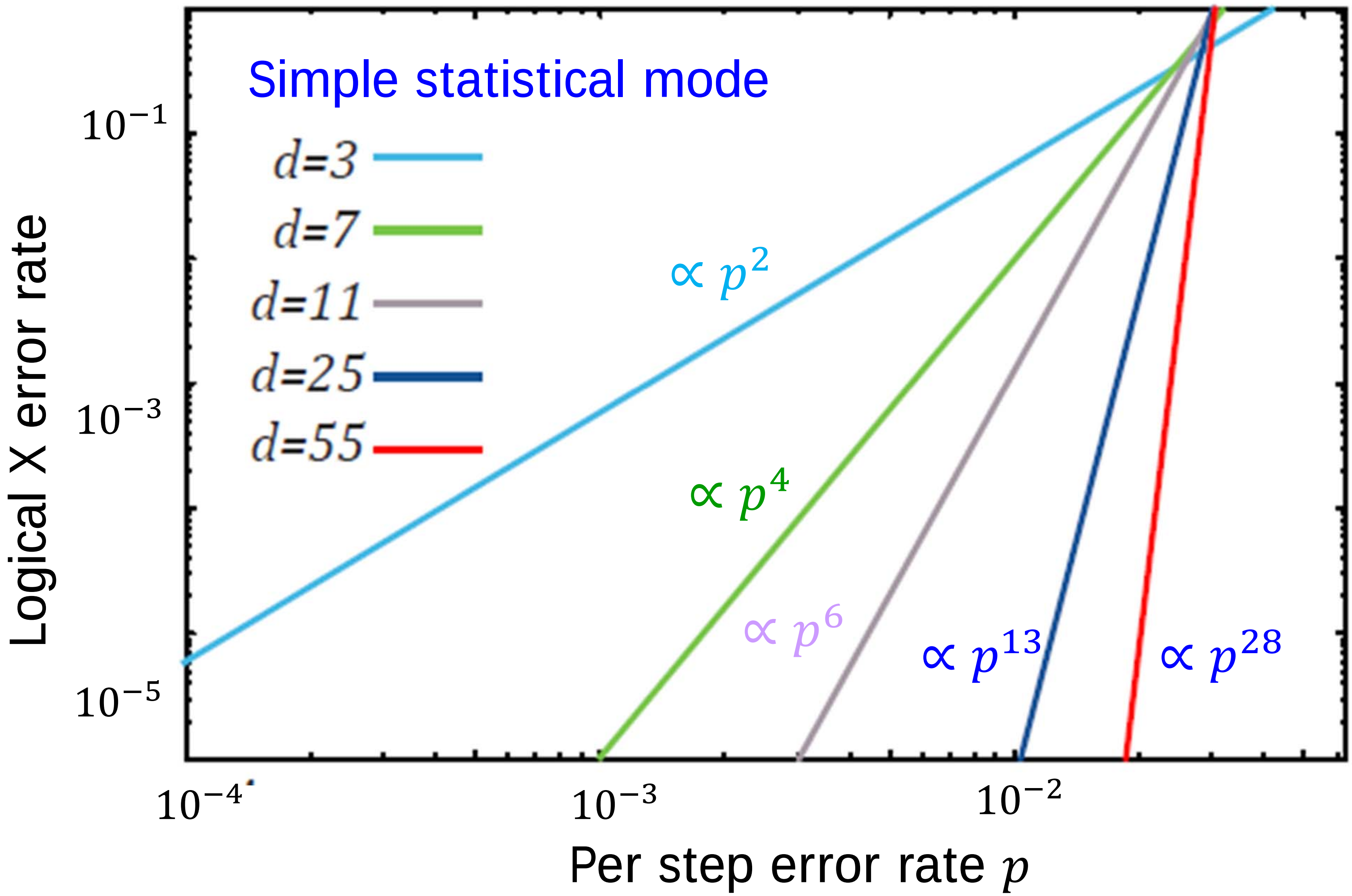


$$X_L = X_1 X_2 X_3 X_4 X_5$$

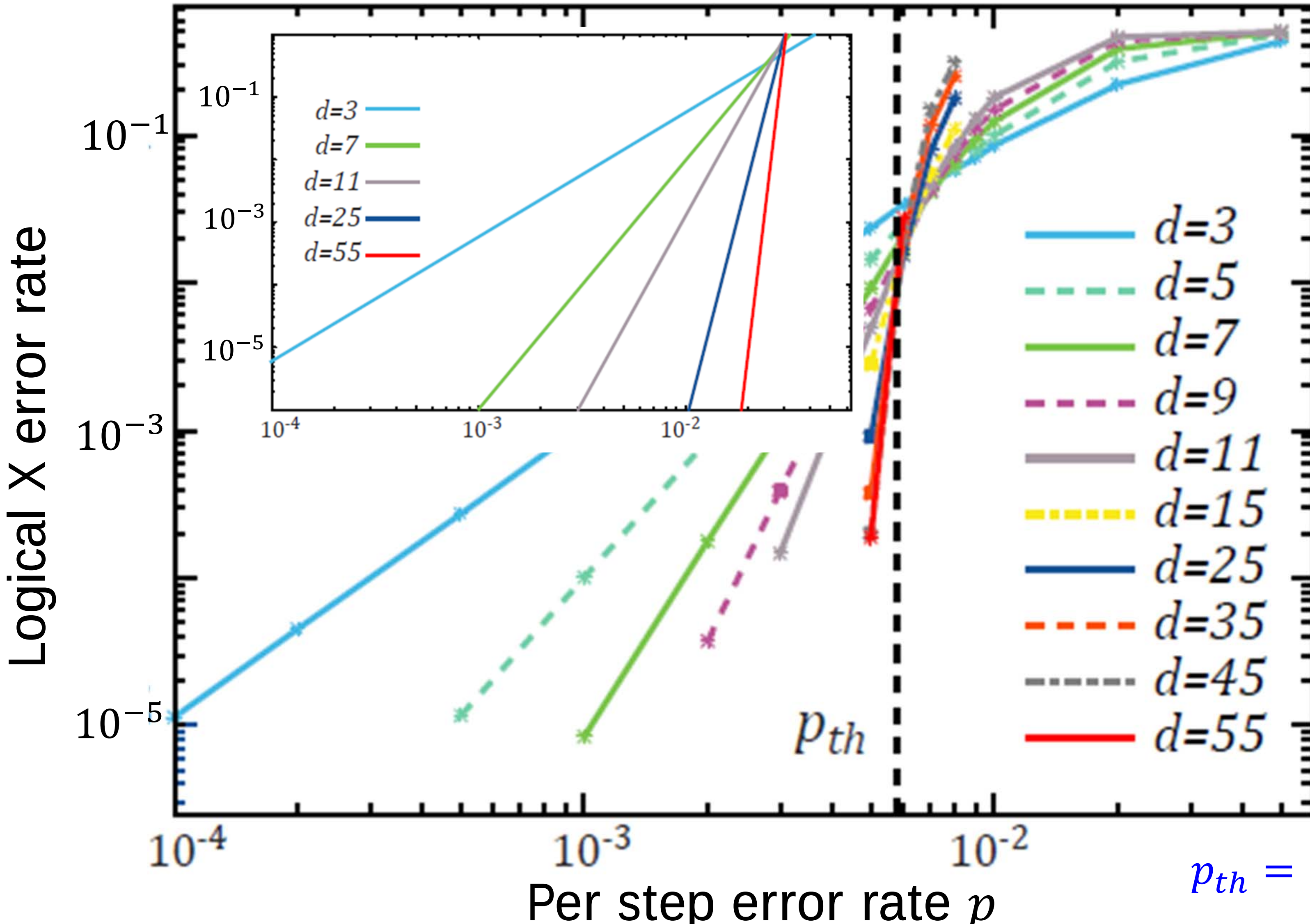
- Chain with distance $d = 5$
- Most common mis-identifications occur for $(d+1)/2$ same-cycle qubit errors being identified as $(d-1)/2$ same-cycle errors
- Here a 3-qubit error erroneously identified as a 2-qubit error (note these are **complementary**)
- This causes a **logical error**
- This occurs at a rate $\propto p^{(d+1)/2}$

Bigger $d \Rightarrow$ smaller logical error rate

Errors & misidentification



Errors & misidentification



$p_{th} = 0.57\%$

Summary of today's lecture

- A $d \times d$ array of data & measure qubits forms a logical qubit
- Sequence of X and Z stabilizer measurements detects X, Y or Z errors (maximum one per qubit per cycle)
- Spatial pattern of **changed** stabilizer outcomes uniquely identifies location & type of error (if not too many!)
- Stabilizer measurement errors identified by **non-repeatability in time**
- Physical qubit error rate p gives logical error rate $\propto p^{(d+1)/2}$ (if below threshold ~0.57%)
- A $d = 5$ array takes $p = 0.05\%$ to $\sim 99.998\%$ fidelity
- A $d = 9$ array takes $p = 0.05\%$ to $\sim 99.999995\%$ fidelity

Slides for Capri School: Surface Codes

Andrew N Cleland

Capri April 24-28 2017

Lecture 3: Logical Qubits

- Forming logical qubits
- Initialization, measurement, errors
- Logical qubit operations
- Moving & braiding qubits
- Hadamard, S and T gates
- Estimates for sizes

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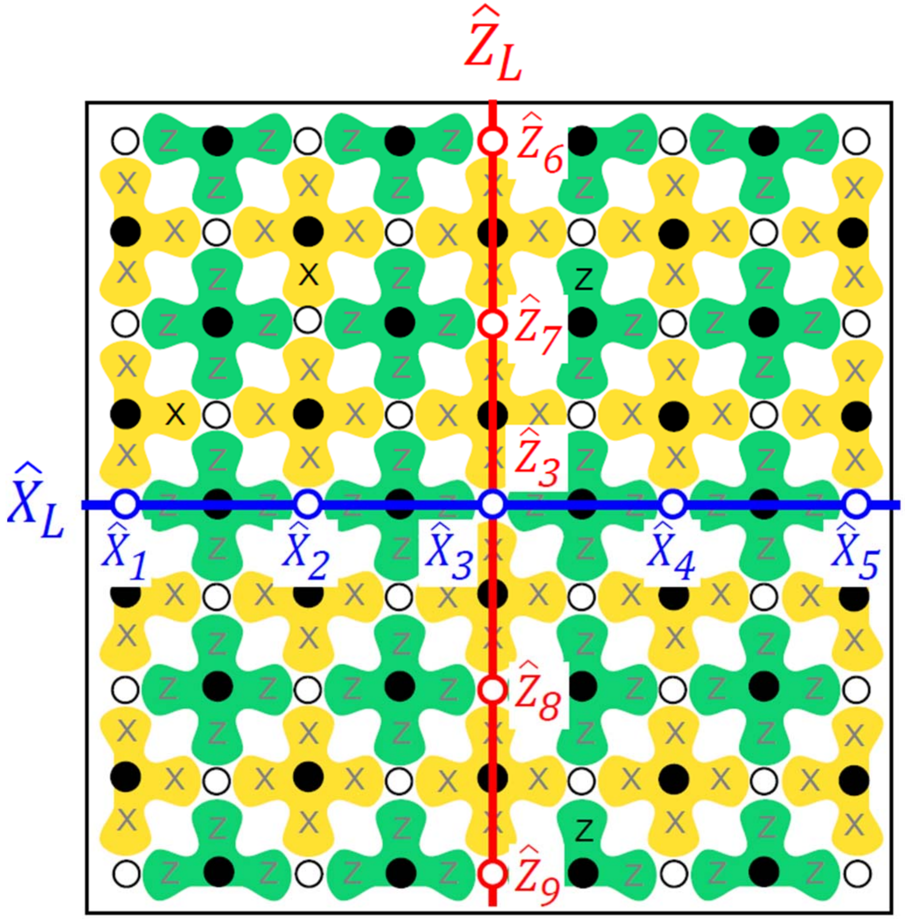
Surface code logical operators



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Same properties as X_L

$$X_L Z_L = X_1 X_2 X_3 X_4 X_5 \times Z_6 Z_7 Z_3 Z_8 Z_9$$
$$= -Z_6 Z_7 Z_3 Z_8 Z_9 \times X_1 X_2 X_3 X_4 X_5$$
$$= -Z_L X_L$$

- X_L and Z_L anti-commute, as desired

$$X_L^2 = Z_L^2 = I \qquad Y_L = i X_L Z_L$$

- All required properties

Summary of Tuesday's lecture

- A $d \times d$ array of data & measure qubits forms a logical qubit
- Sequence of X and Z stabilizer measurements detects X, Y or Z errors (maximum one per qubit per cycle)
- Spatial pattern of **changed** stabilizer outcomes uniquely identifies location & type of error (if not too many!)
- Stabilizer measurement errors identified by **non-repeatability in time**
- Physical qubit error rate p gives logical error rate $\propto p^{(d+1)/2}$ (if $p < 0.57\%$)
- A $d = 5$ array takes $p = 0.05\%$ to $\sim 99.998\%$ fidelity
- A $d = 9$ array takes $p = 0.05\%$ to $\sim 99.999995\%$ fidelity
- This is a memory qubit only – no logic capability!

Logical qubits: Z-cut qubit

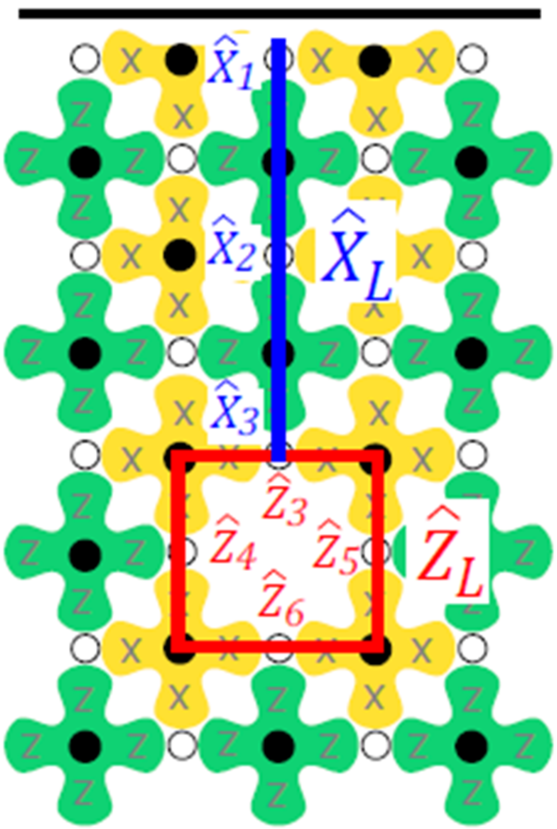
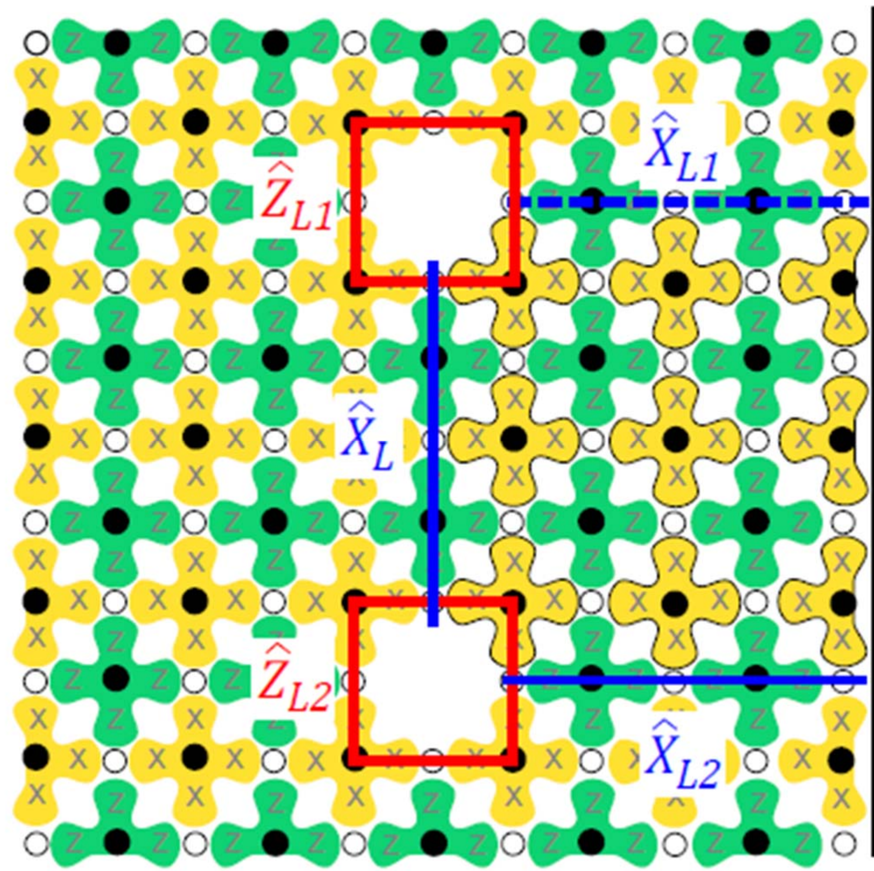


fig 9

Edge logical Z-cut qubit:
Turn off one Z stabilizer



Double Z-cut qubit:
Turn off two Z stabilizers

fig 10

Logical qubits: Double Z,X-cut qubits

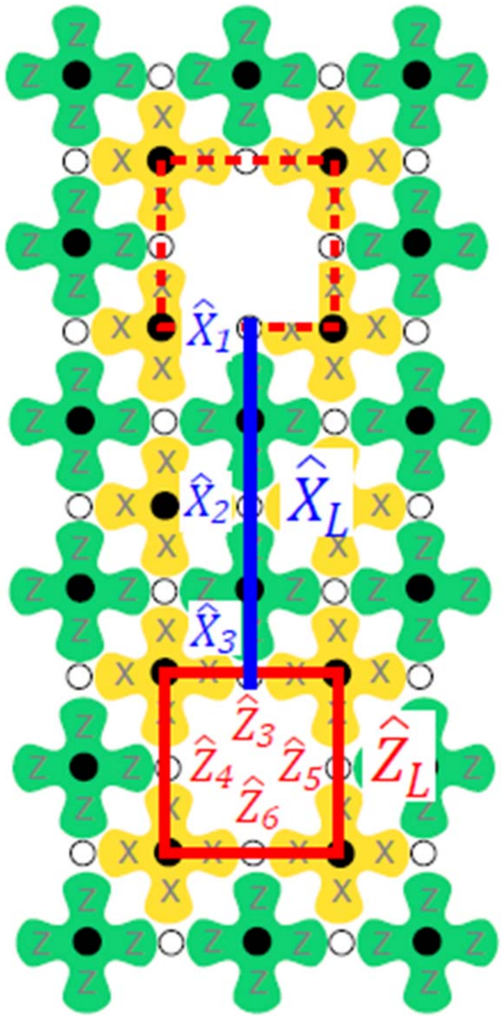
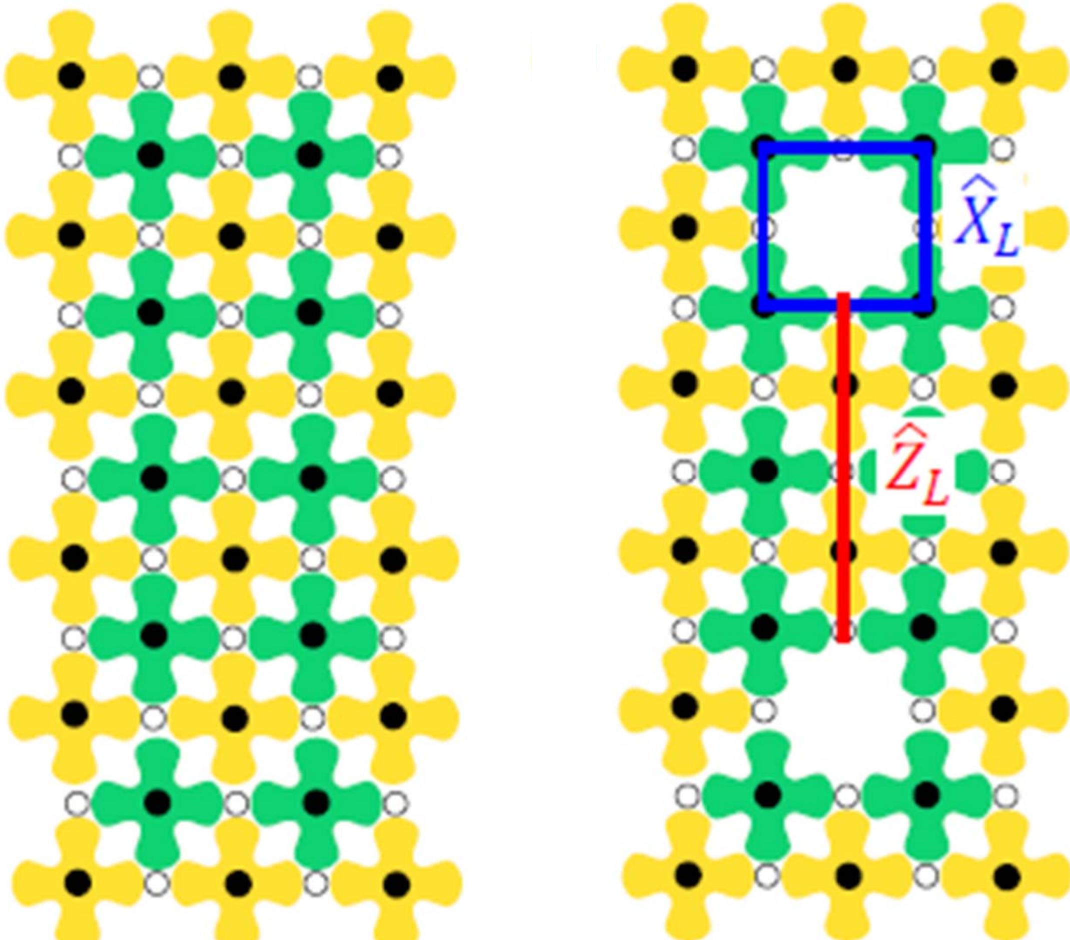


fig 11

Initializing a Logical Qubit

Initializing an X-cut qubit in an X eigenstate



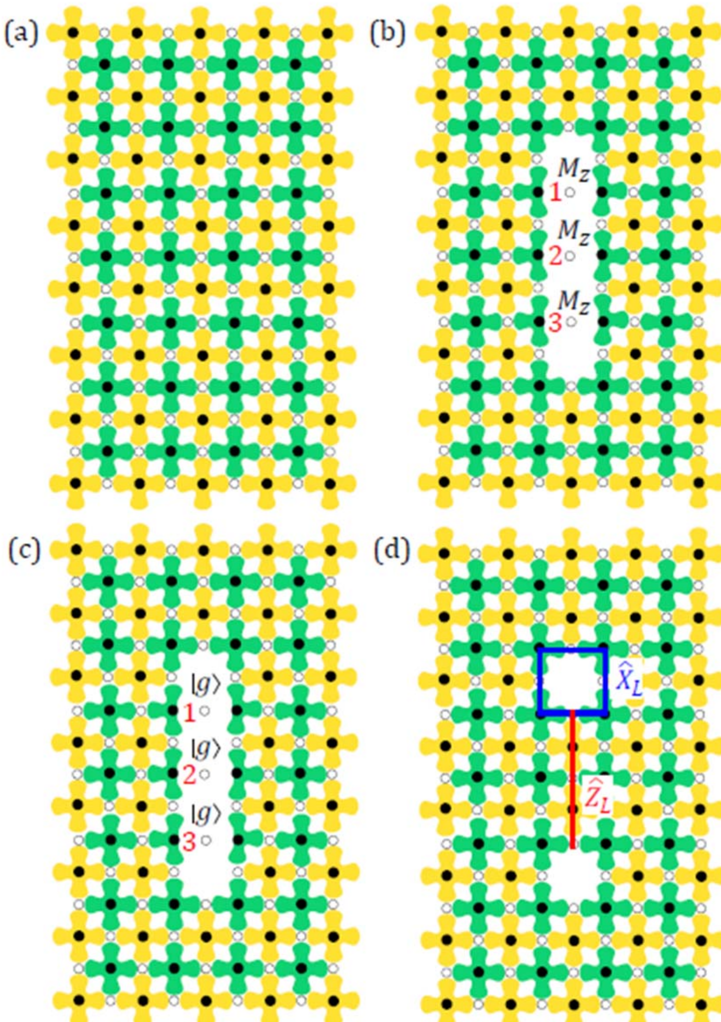
Logical X value is that of X stabilizer prior to forming cut

To measure: Simply turn measure X qubit back on and report value

fig 13

Logical qubits: Initialize & Measure

Initializing X-cut along Z



Measuring X-cut along Z

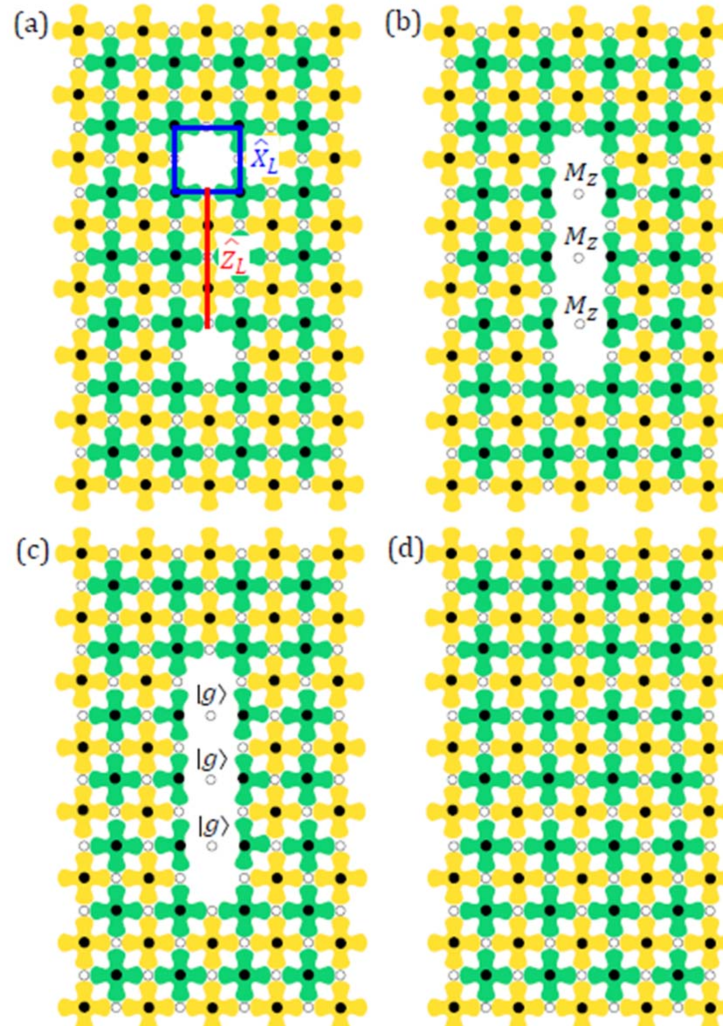


fig 14

Logical qubits: Moving a qubit

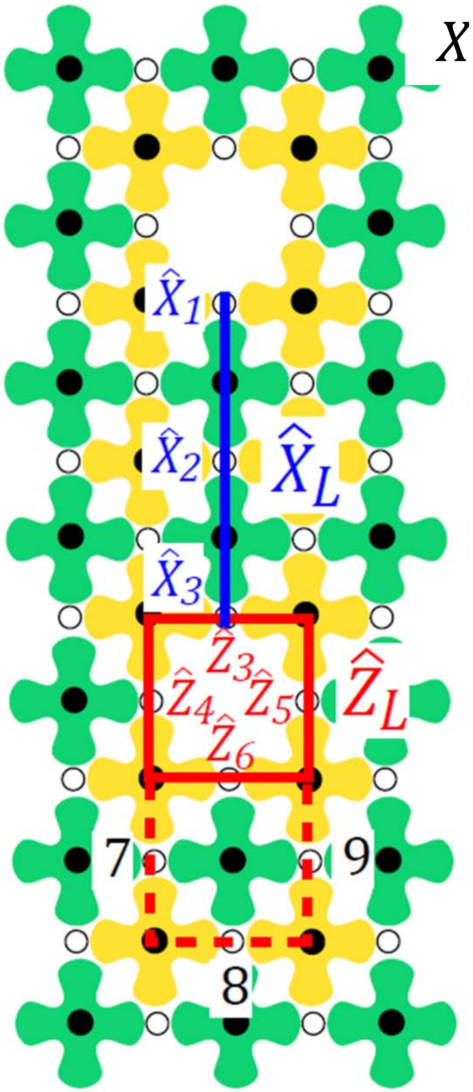


fig 17

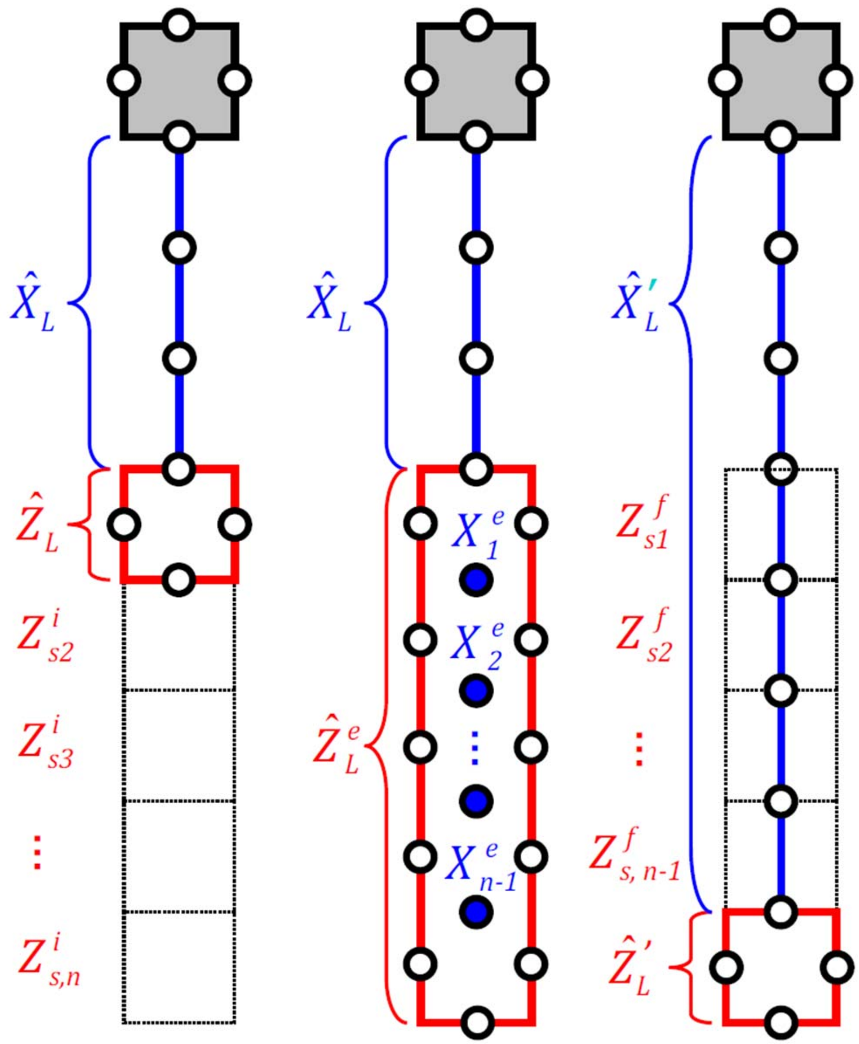
X

$Z_6 Z_7 Z_8 Z_9$

$_9 \times Z_{stab}$

Z_{stab}

Logical qubits: Moving a qubit



Extend X_L to X'_L by multiplying by X values of intermediate data qubits

Extend Z_L to Z^e_L by multiplying by intermediate Z stabilizer values

Convert Z^e_L to Z'_L by multiplying by final Z stabilizer values

fig 18

Logical qubits: Empty braid

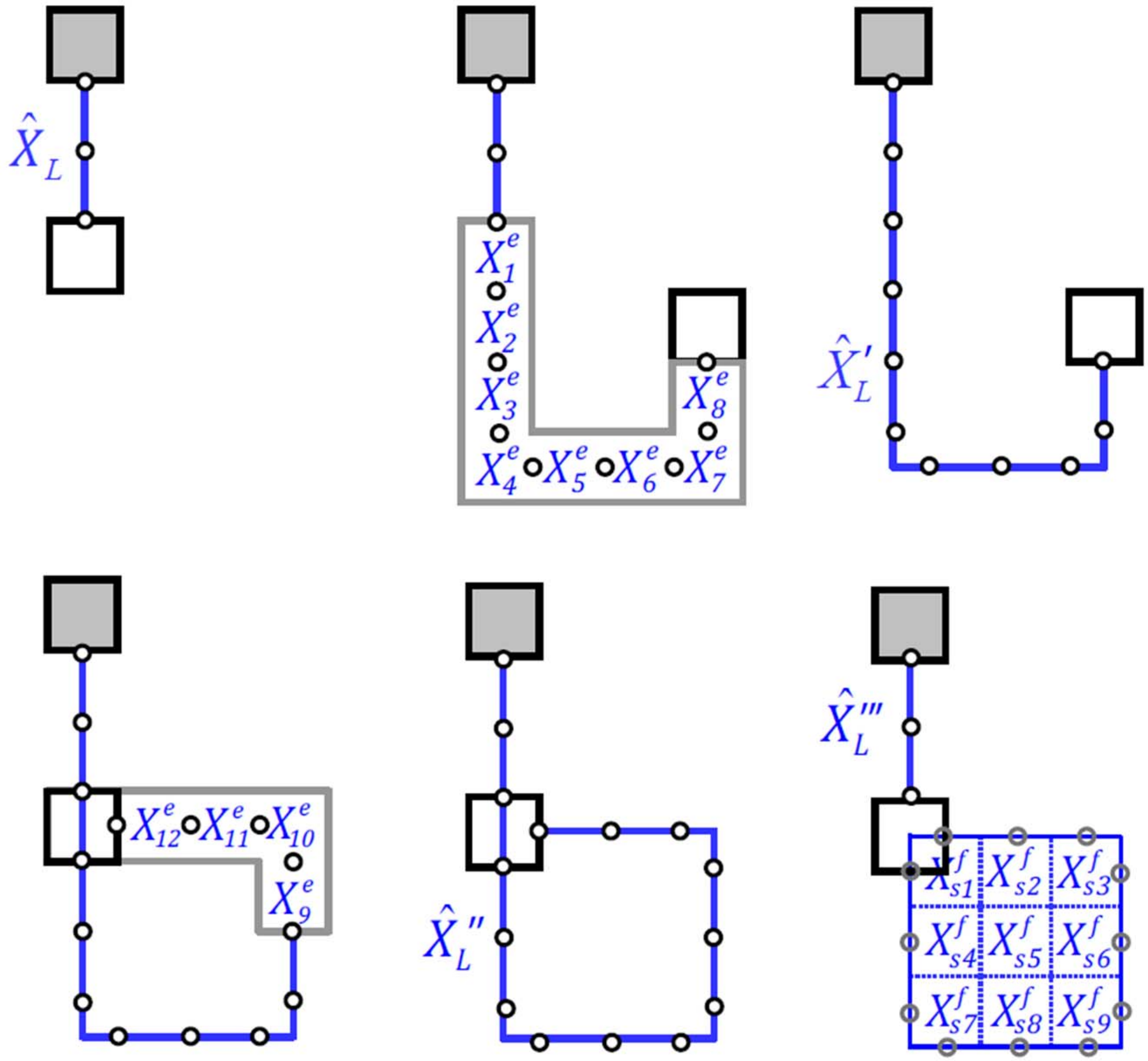
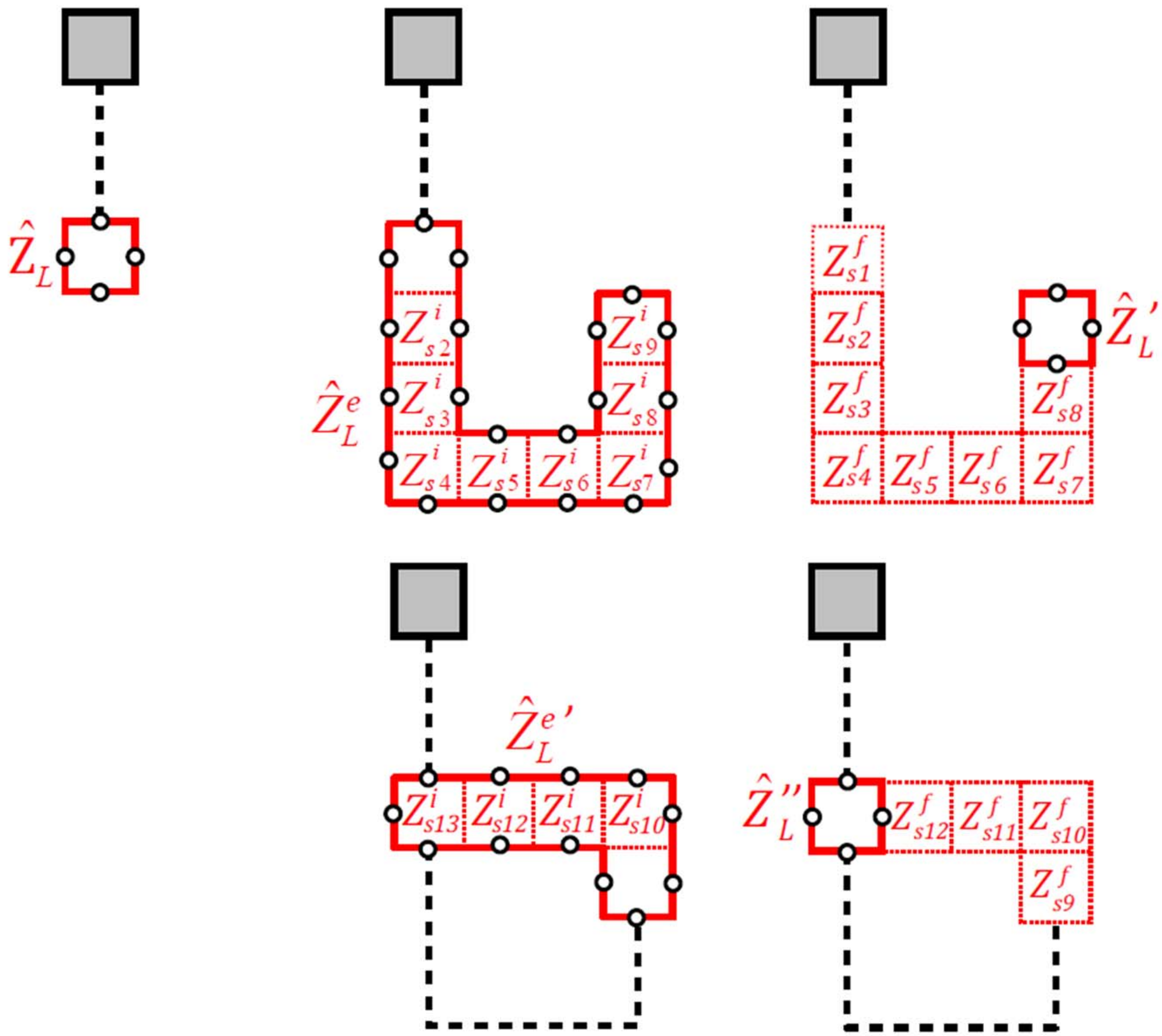


fig 19

Final $X_L''' = \pm X_L$
Sign determined by enclosed stabilizer values $X^f_{s1} \ldots X^f_{s9}$

Logical qubits: Empty braid



Final $Z_L''' = \pm Z_L$
Sign determined by intermediate Z stabilizers in move

fig 20

CNOT: Operator picture

CNOT gate: if control qubit in $|e\rangle$, flip (NOT) target qubit

- Want to validate an operation C as a CNOT

$$C|\psi\rangle = |\psi'\rangle$$

$$\langle gg|C|gg\rangle = 1$$

$$\langle gg|C|ge\rangle = 0$$

...

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

4 basis states

4^2=16 parameters

- Verifying the operation performs a CNOT can be done by finding how pre-operation basis states are mapped to post-operation basis states
- Alternative is to check that all operators are transformed correctly (Heisenberg picture)
- All one-qubit operators are linear combinations of I, X, Y, Z
- All two-qubit operators are linear combinations of $I \otimes I, I \otimes X, I \otimes Y, I \otimes Z$ ... ($4^2 = 16$ combinations)
- Check $C^\dagger (U \otimes V) C = P \otimes Q$ for all combinations U, V

CNOT: Operator picture

However only four operator combinations are independent:

$$I \otimes X \Rightarrow I \otimes X \qquad I \otimes Z \Rightarrow Z \otimes Z$$

$$X \otimes I \Rightarrow X \otimes X \qquad Z \otimes I \Rightarrow Z \otimes I$$

For example $C^\dagger(X \otimes Z)C = C^\dagger(X \otimes I)(I \otimes Z)C$

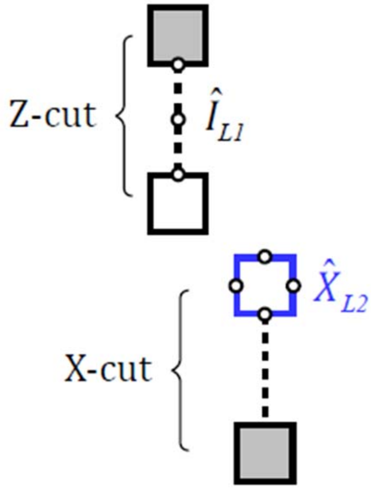
$$= C^\dagger\big(X \otimes I)C^\dagger C(I \otimes Z\big)C$$

$$= (X \otimes X)(Z \otimes Z)$$

$$= XZ \otimes XZ = -Y \otimes Y$$

Hence only need to check four operator transformations to test a proposed CNOT operation...

Braiding $I \otimes X$



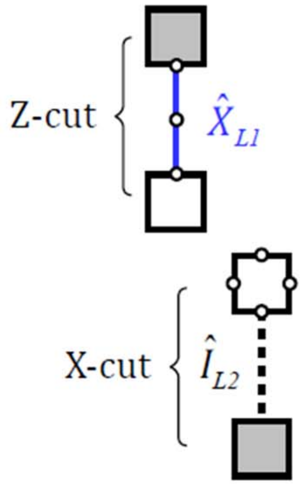
Final step in braid:

$$I_{L1} \otimes X_{L2} = I_{L1} \otimes X_{L2}$$

(first CNOT identity)

fig 22

Braiding $X \otimes I$



Final step in braid:

$X''_{L1} \otimes I_{L2} = X'''_{L1} \otimes X_{L2}$

(second CNOT identity)

fig 21

Braiding $I \otimes Z$ and $Z \otimes I$

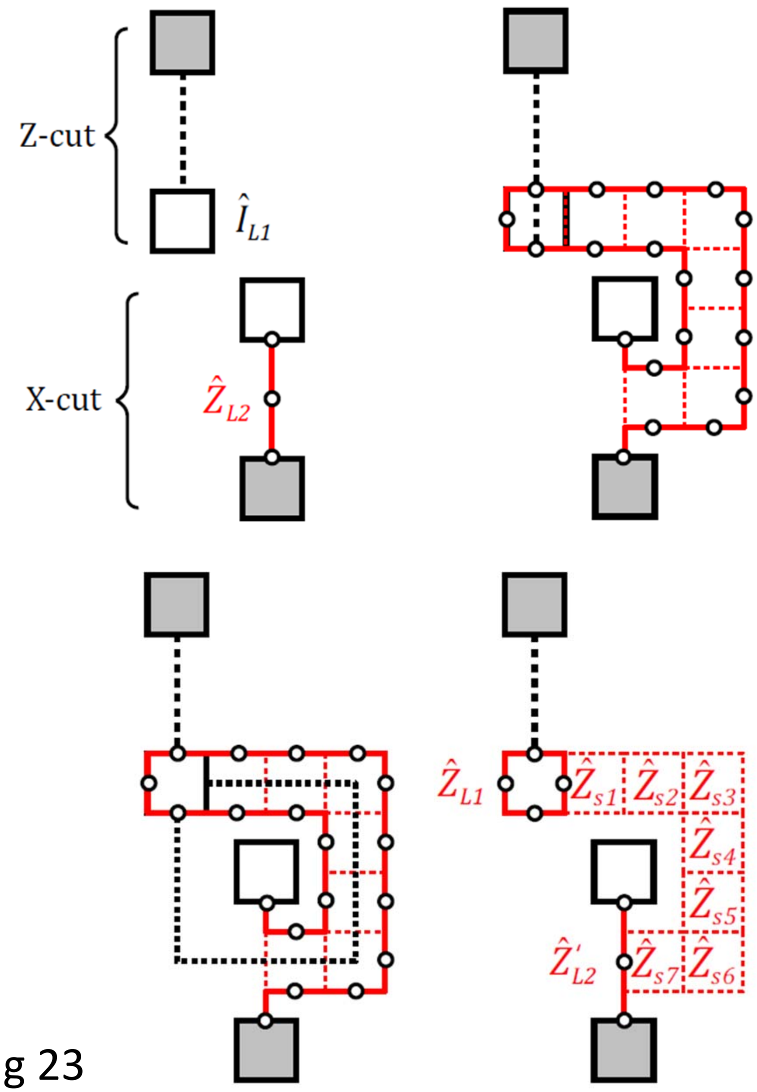


fig 23

$I \otimes Z$:

Final step:

Turn on all Z stabilizers

Restores Z'_{L2} but generates Z_{L2}

$$I \otimes Z \Rightarrow Z \otimes Z$$

$Z \otimes I$:

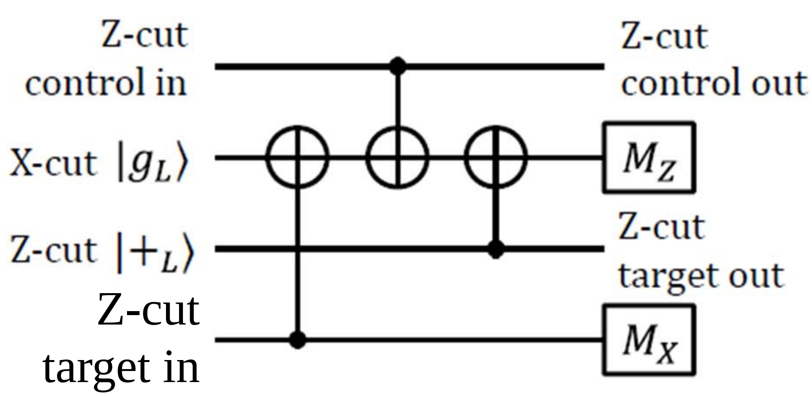
Easy to show that transformation does not change operators

- Braid operation is equivalent to CNOT
- Surface code protection maintained

Extending to all qubit cuts

- All operations involve Z-cut control qubit and X-cut target qubit
- Quantum circuits extend to Z-Z and X-X with 2 ancillas

Z control, Z target



X control, X target

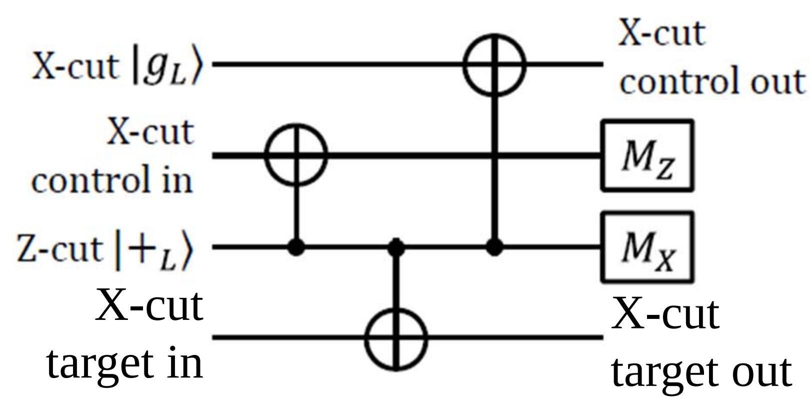


fig 24

Logical Hadamard

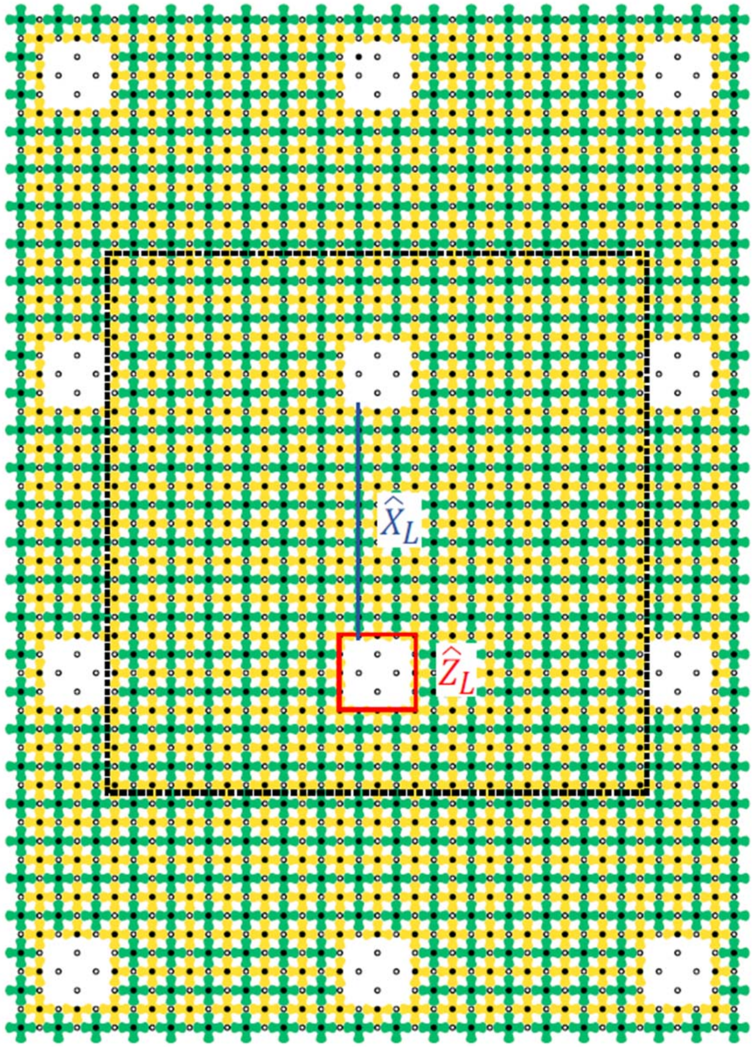


fig 26

Logical Hadamard

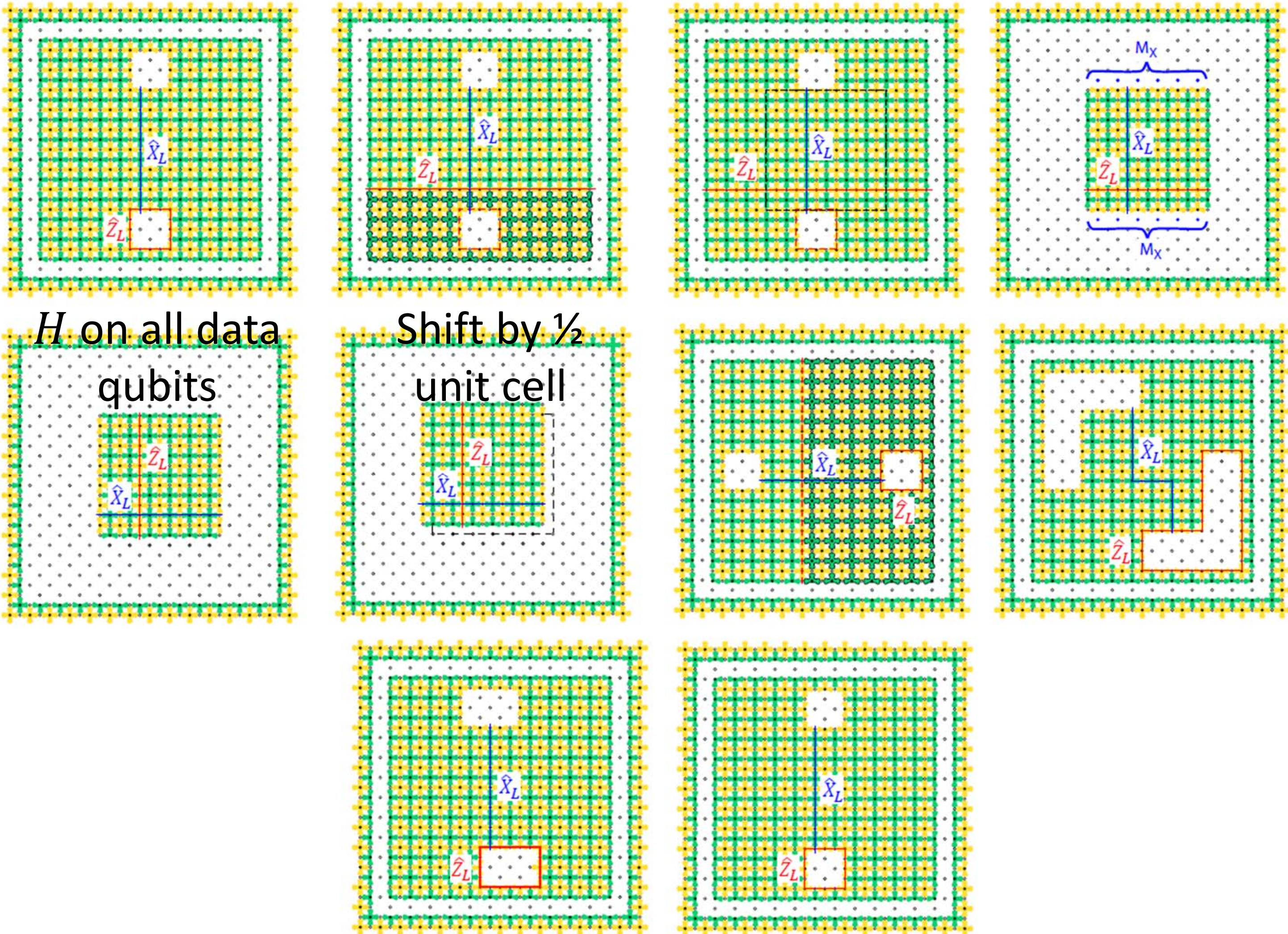
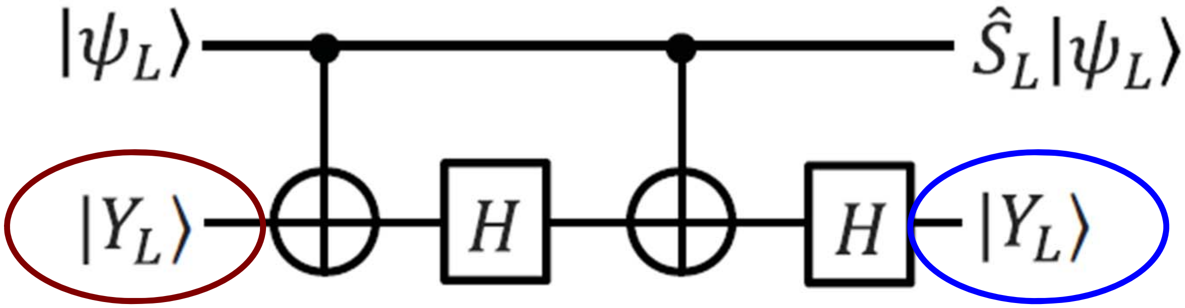


fig 27&28

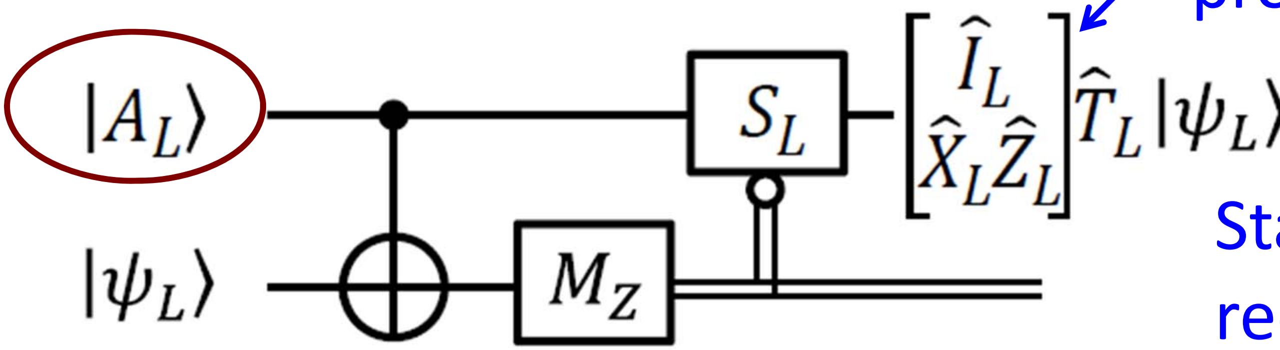
Logical S and T gates



State is recycled!

$$|Y_L\rangle = (|g_L\rangle + i|e_L\rangle)/\sqrt{2}$$

Result is probabilistic

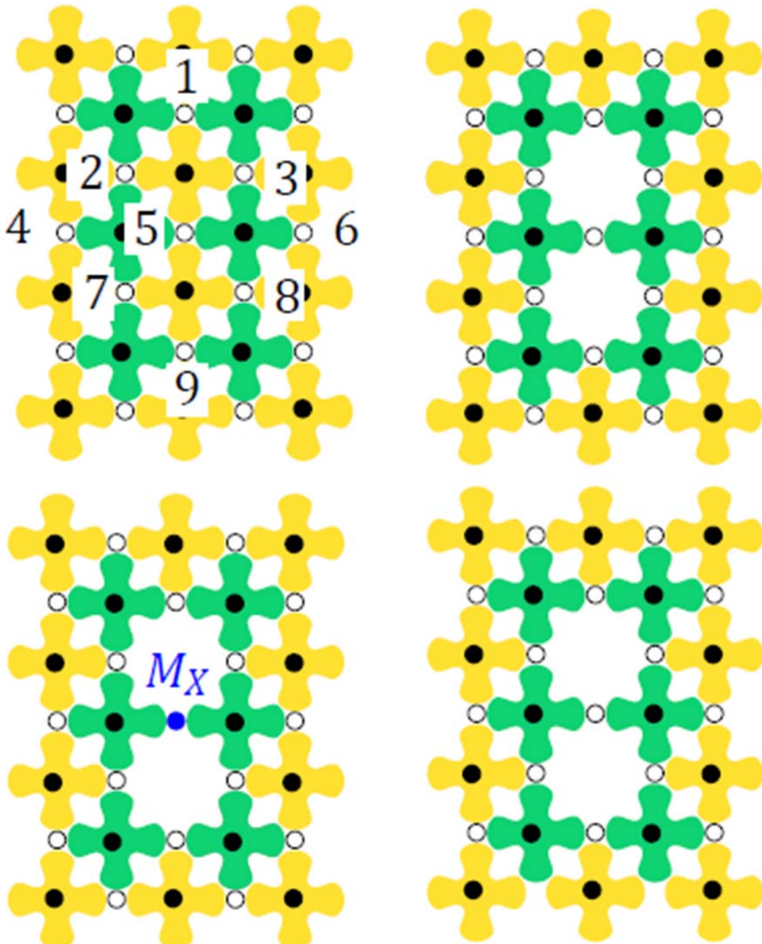


State is **not** recycled ☹

$$|A_L\rangle = (|g_L\rangle + e^{i\pi/4}|e_L\rangle)/\sqrt{2}$$

fig 29

"Short" qubit

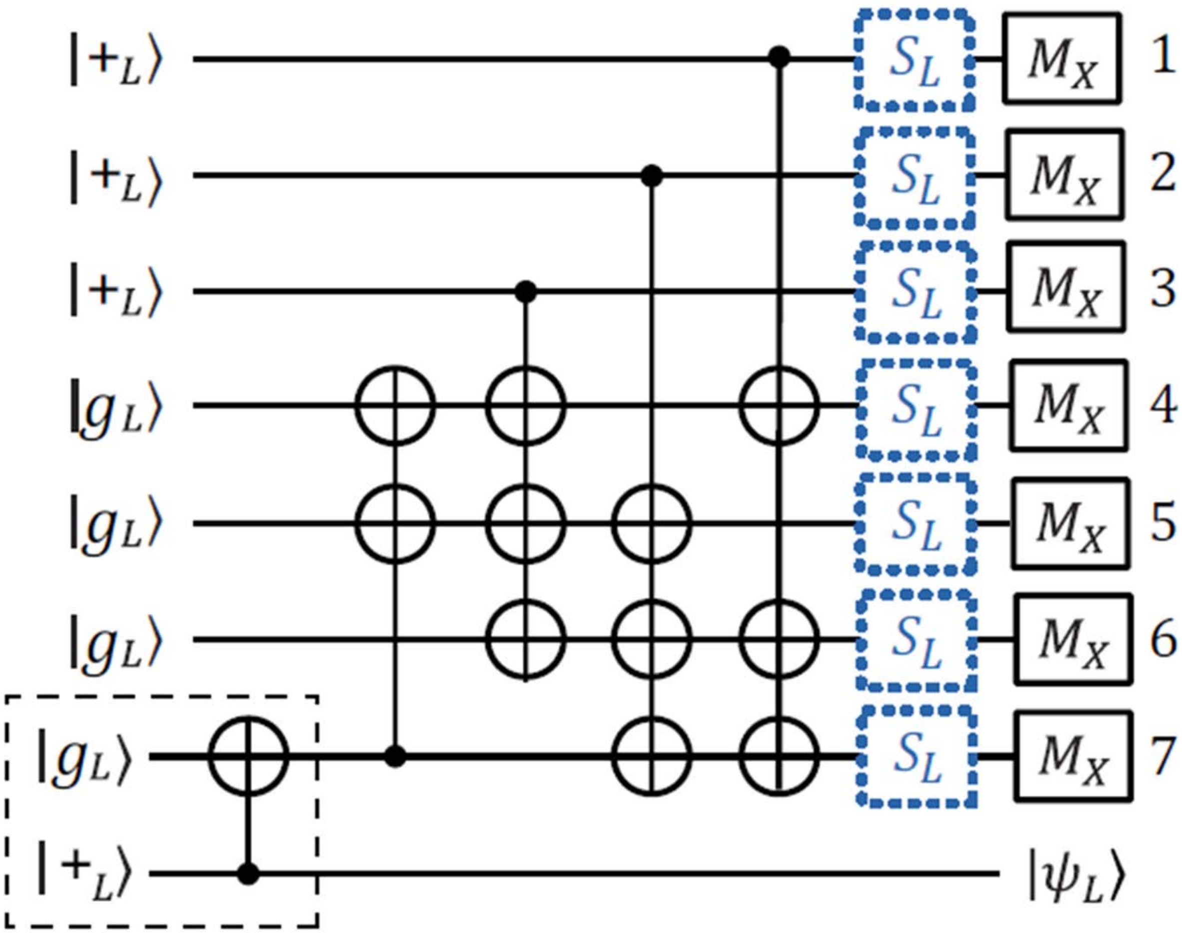


$(|g_L\rangle + e^{i\theta}|e_L\rangle)/\sqrt{2}$

- Can inject arbitrary state $(|g_L\rangle + e^{i\theta}|e_L\rangle)/\sqrt{2}$
- Amplitudes & phase limited by control electronics
- Cannot expect better than $\sim p$ precision
- How to use in logical qubit circuit?

fig 31

Magic "Y" state distillation



- Seven approximate $|Y_L\rangle$ states used in seven S_L gates
- Output $|\psi_L\rangle$ is purified $|Y_L\rangle$ state
- Used as input for next round of purification

fig 32

Magic "A" state distillation

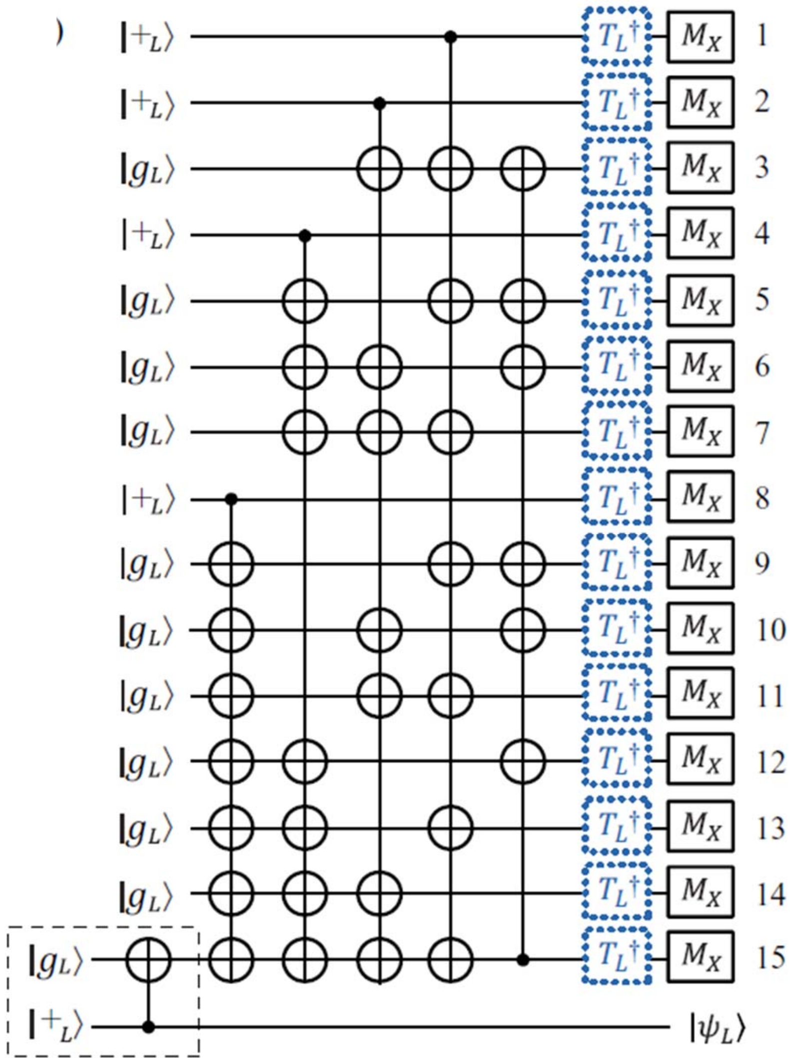


fig 33

- Fifteen approximate $|A_L\rangle$ states used in 15 T_L gates
- Output $|\psi_L\rangle$ is purified $|A_L\rangle$ state
- Used as input for next round of purification
- Four rounds requires 15^4 =50,625 initial $|A_L\rangle$ states to generate one purified state
- State not recycled!

Surface Code

- Assume error rate 1/10th threshold (99.95% fidelity)

Logical memory qubit from array of physical qubits

- × 1,000 smaller error rate: ~600 physical qubits
- × 1,000,000 smaller error rate: ~2,000 physical qubits
- × 1,000,000,000 smaller rate: ~4,500 physical qubits

Circuit to demonstrate topological CNOT:

- With × 1,000 smaller error rate: ~1,800 physical qubits

Prime factoring with Shor's algorithm:

- Factor a 15 bit number (10^5): ~40,000,000 qubits
- Factor a 2000 bit number (10^{600}): ~1,000,000,000 qubits

~99% of factoring computer is "A-state factories"

Fowler et al. (PRA, 2012)