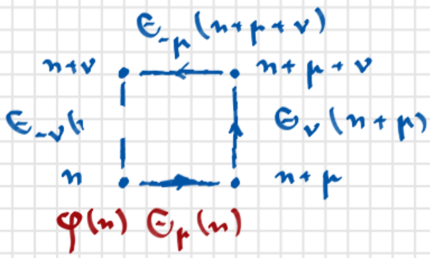
Physical foundations of surface codes

Motivation: Progress in the field driven by two sources of inspiration: Information theory and stat. mech. / cond. mat. analogies. (confinement, string excitations, electric and magnetic excitations, string cond.) Here: introduce surface codes from cond. perspective.

... back into 1970s ... consider xy-model on d-dimensional lattice

- $\Theta_\mu(n) = \varphi(n+\mu) - \varphi(n)$



- $\Theta_\mu(n) \in [0, 2\pi]$
- $\Theta_{-\mu}(n+\mu) = -\Theta_\mu(n)$

$$S[\Theta] = K \sum_{n,\mu} \cos(\Theta_\mu(n)) \qquad K = J/T$$

Model has global symmetry: $\Theta_\mu(n) \to \Theta_\mu(n) + \chi$

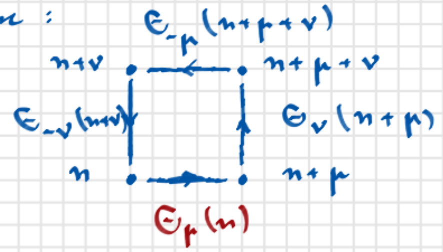
$\rightsquigarrow$ In dimensions $d > 2$: spontaneous symmetry breaking below critical temperature $\langle e^{i\varphi(n)} \rangle \neq 0$.

Q: What happens if we promote the symmetry to a local symmetry $\rightsquigarrow$ Lattice gauge theory? Note: It is often more productive to think of 'gauge symmetry' as presence of redundant degrees of freedom whose removal generates correlations.

Lattice gauge theory in a nutshell (Review: Kogut, 79)

Example: $U(1)$ lattice gauge theory on a 4d lattice – discrete Electrodynamics.

Consider lattice structure:



- $\Theta_\mu(n) \in [0, 2\pi]$
- $\Theta_{-\mu}(n+\mu) = -\Theta_\mu(n)$

Goal: Define theory of $\{\Theta_\mu(n)\}$ that contains a local gauge symmetry defined at the sites, n, of the lattice.

Def.: $$S[\Theta] = \frac{1}{2g^2} \sum_{\text{plaquettes}} \prod_{\circlearrowleft \text{ links}} e^{i\Theta_\mu(n)}$$

Action is invariant under local transformation $\Theta_\mu(n) \to \Theta_\mu(n) + \chi(n) - \chi(n+\mu)$ where $\chi(n) \in [0, 2\pi]$ is a lattice function

Relation to continuum electrodynamics: Define a: lattice spacing, and $\Theta_\mu(n) \equiv a\, g\, A_\mu(n)$. For small g and smooth A:

$$\sum \prod_{\circlearrowleft} e^{i\Theta_\mu(n)} = \sum \prod_{\circlearrowleft} e^{iagA_\mu(n)} = \sum e^{iag\sum_{\circlearrowleft} A_\mu(n)} \simeq$$

$$\simeq iag \sum \sum_{\circlearrowleft} A_\mu(n) - \frac{1}{2} a^2 g^2 \sum \Big(\sum_{\circlearrowleft} A_\mu(n) \Big)^2$$

$$\left[\begin{aligned} \sum_{\circlearrowleft} A_\mu(n) &= A_\mu(n) + A_\nu(n + a e_\mu) - A_\mu(n + a e_\nu) - A_\nu(n) \simeq \\ &\simeq -a\partial_\nu A_\mu(n) + a\partial_\mu A_\nu(n) \end{aligned} \right.$$

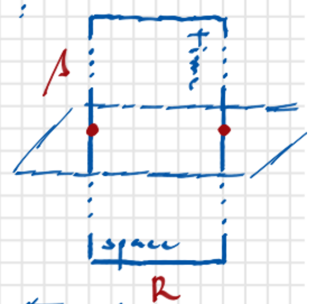
$$\simeq -\frac{1}{4} \int d^4x \, (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$$

$\rightsquigarrow$ the (euclidean) action of electrodynamics. Gauge transformation: $A_\mu \to A_\mu + \partial_\mu \chi$

Q: How does the discrete theory differ from the continuum limit?

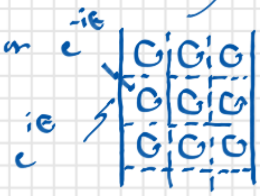
Elizur theorem: gauge non-invariant observables (such as $e^{i\varepsilon_\mu(n)}$) have vanishing expectation values.

Good gauge invariant observables: $e^{-i\sum_C \varepsilon_\mu(n)}$ where the sum is over closed loop (Wegner - Wilson loops). An interesting reference loop:



Continuum formulation of loop: $L = \exp\left(-i\oint A_\mu dx^\mu\right)$. Closed unit current loop. At '•': $j_0 = \pm 1$, i.e. loop describes two unit charges separated by distance R.

Strong coupling: $g \gg 1$, large fluctuations of phase variables. Need to fill area of loop with plaquette operators to effect phase cancellation

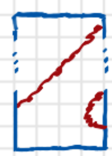


$$\langle L\rangle \sim \left(\frac{1}{2g^2}\right)^{R\cdot\beta} \sim e^{2\ln g\, R\cdot\beta} \qquad \text{'area law'}$$

Free energy of charge system grows linearly in distance strong confinement.

Weak coupling: Work in continuum representation: $\langle A_\mu(x)\, A_\nu(x')\rangle = \delta_{\mu\nu}\frac{1}{2\pi^2}\frac{1}{|x-x'|^2}$ for $|x-x'| > a$ in Feynman gauge ($\Rightarrow \Box A_\mu = 0$ as equations of motion).

$$\langle L\rangle = \left\langle e^{ig\oint A_\mu dx^\mu}\right\rangle = e^{-\frac{1}{2}g^2\oint\oint dx^\mu dx'^\nu \langle A_\mu(x)\, A_\nu(x')\rangle} =$$



$$= e^{-\frac{1}{2}g^2 \oint\oint_{\text{parallel}} dx^\mu dx'^\mu \frac{1}{2\pi^2}\frac{1}{|x-x'|^2}} = \dots \text{ some work } \dots =$$

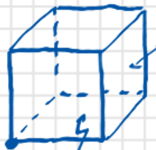
$$\underset{\beta \gg R}{=} e^{-\frac{1}{2}g^2 \underbrace{c}_{\text{const.}} P + \frac{g^2}{2\pi}\frac{\beta}{R} + \frac{2g^2}{\pi^2}\ln(R/a)} \qquad P = 2(T+R) = \text{loop perimeter}$$

Interpret $\beta = T^{-1}$ and $\langle L\rangle = Z(g)/Z(0) \sim -\beta^{-1}\ln\langle L\rangle = +F(g) - F(0)$

For $\beta \to \infty$: $F(g) - F(0) = \frac{g^2}{2\pi}\frac{1}{R} + \text{const.}$ Deconfinement (+ Coulomb law) if $g = e$ is identified.

Q: Consider Z as imaginary time field integral of lattice QED. What is its Hamiltonian

A: Take continuum limit in t ~ direction. Fix gauge $A_0 = 0$, or $\Theta_0(n) = 0$



$g_\tau \ll g_s$

Spatio-temporal plaquette: $\frac{1}{2g_\tau^2} e^{i(\Theta_i(x,\tau) - \Theta_i(x,\tau+a_\tau))}$

$$\to -\frac{1}{2g_\tau^2} a_\tau^2 \left(\partial_\tau \Theta_i(x,\tau)\right)^2$$

$$S = \frac{a_\tau}{2g_\tau^2}\int d\tau \sum_x \partial_\tau \Theta_i \partial_\tau \Theta_i + \frac{1}{2g_s^2 a_\tau}\int d\tau \sum_x \prod_{\square,\,\text{spatial}} e^{i\Theta_i(x)}$$

Def: $a_\tau g_\tau^{-2} \equiv g^{-2} a \qquad a_\tau^{-1} g_s^{-2} \equiv g^{-2}$

$$= \frac{a}{2g^2}\int d\tau \sum_x \partial_\tau \Theta_i \partial_\tau \Theta_i + \frac{1}{2g^2}\int d\tau \sum_x \prod_{\square} e^{i\Theta_i(x)} \equiv \int d\tau\, L(\Theta,\dot\Theta)$$

Canonical momenta of variables $\Theta_i(x)$: $\pi_i(x) \equiv \frac{\partial L}{\partial \partial_\tau \Theta_i(x)} = a g^{-2} \partial_\tau \Theta_i(x)$ (*)

Hamiltonian action:

$$S = \int d\tau \sum_x \left(\pi_i \partial_\tau \Theta_i(x) + \frac{1}{2}\left(\frac{g^2}{a}\pi_i^2(x) + \frac{1}{g^2}\prod_{\square} e^{i\Theta_i(x)}\right)\right)$$

$$= \int d\tau \sum_x \left(\pi(x)\partial_\tau \Theta_i(x) + \mathcal{H}(\Theta_i,\pi_i)\right)$$

Hamiltonian density

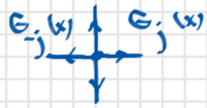
recall $\Theta_i = a g A_i$. In gauge $A_0 = 0$: $\partial_\tau A_i(x) = E_i(x)$, electric field.

$$\pi_i(x) \overset{(*)}{=} a^2 g^{-1} \partial_\tau A_i(x) = a^2 g^{-1} E_i(x)$$

$\leadsto$ *Hamiltonian:* $H = \sum_x \mathcal{H} \to \frac{a^3}{2} \sum_x E_i^2(x) + \frac{1}{2g^2} \sum \prod_{\circlearrowleft} e^{i\Theta_i(x)} \xrightarrow[g\to 0]{} \frac{1}{2} \int d^3x \left(E_i^2 + B_i^2\right)$

Generators of gauge transformations: $[\Theta_i(x), \pi_j(y)] = -i\,\delta_{ij}\,\delta_{xy}$

$\leadsto \; e^{i \sum_{\pm j} \pi_j(x)\,\chi}$ *shifts all*



$\Theta_{\pm j}(x) \to \Theta_{\pm j}(x) + \chi$

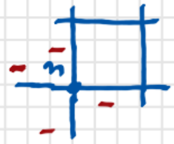
$\Theta := e^{i \sum_{x,j} \pi_j(x)\,\chi(x)}$ *generates gauge trafo:* $\Theta\, \Theta_j(x)\, \Theta^{-1} = \Theta_j(x) - \chi(x) + \chi(x+j)$

$\mathbb{Z}_2$ - Lattice gauge theory (Wegner 71)

Replace $\epsilon_\mu(n) \in \{+1,-1\}$ by binary state variables.

$$S[\epsilon] = \frac{1}{2g}\sum_{\square}\prod_{\circlearrowleft}\epsilon_\mu(n)$$

Local $\mathbb{Z}_2$ gauge invariance $\epsilon_\mu(n) \to -\epsilon_\mu(n)$



Expect two phases: $g \gg 1$ strong fluctuation / confinement
$g \ll 1$ weak " / deconfinement

Explore situation in equivalent $d-1$ dimensional quantum system. Here $d=3$ classical $\to$ $d=2$ quantum.

$U(1)$ classical

state variable, $\epsilon_i(n)$

quantum

operator $\hat{\epsilon}_i(n) \sim A_i(n)$; $e^{iag\hat{A}_i(n)}$

conjugate $\hat{E}_i(n)$

generator gauge trafo: $e^{i\frac{a^2}{g}\sum_i \hat{E}_i(n)}$

Hamiltonian: $\hat{H} = \frac{a^2}{2}\sum_i \hat{E}_i^2 + \frac{1}{2g}\sum_{\square}\prod_{\circlearrowleft} e^{i\hat{\epsilon}_i(n)}$

$\mathbb{Z}_2$ state variable $\epsilon_i(n)$

operator $\sigma_{zi}(n) \sim \hat{A}_i(n)$

conjugate $\sigma_{xi}(n) \sim \hat{E}_i(n)$

generator gauge trafo: $\prod_i \sigma_{xi}(n)$

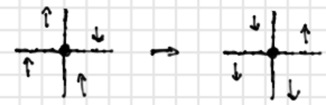
Hamiltonian: $\hat{H} = -\gamma\sum_{i,n}\sigma_{xi}(n) - \lambda\sum_{\square}\prod_{\circlearrowleft}\sigma_{zi}(n)$

$\mathbb{Z}_2$-lattice gauge theory (Wegner 71, review Kogut RMP 79)

- $\mathbb{Z}_2$-degrees of freedom frequently emerging in correlated fermion systems (Senthil & Fisher, PRB 62, 7850 (2000)): $c^\dagger \to e^{i\phi} \cdot f^\dagger = e^{i\phi}(-1)\times(-1)\, f^\dagger$
  $c^\dagger$: fermion; $e^{i\phi}$: charge; $f^\dagger$: neutral fermion
  $\rightsquigarrow$ an emergent 'gauge degree of freedom'
- Formulate $\mathbb{Z}_2$ gauge theory on a lattice. (Here 2d $\square$-lattice for simplicity)

$\mathbb{Z}_2$-d.o.f. $|\pm\rangle_{ij}$ eigenstates of op $\sigma_{z,ij}$.

- local gauge trafo at i acts through $\prod_{\text{n.n. } i} \sigma_{x,ij}$
  $\rightsquigarrow |\pm\rangle_{ij} \to |\mp\rangle_{ij}$



- dynamical players of $\mathbb{Z}_2$ lattice gauge theory
  - gauge field along link $i\to j$: $\sigma_{z,ij}$ $\left(\hat{=}\, e^{iA_{ij}} \text{ in } U(1) \text{ lattice ED}\right)$
  - electric flux through link $i\to j$: $\sigma_{x,ij}^{(*)}$. $\quad \sigma_z\sigma_x\sigma_z = -\sigma_x \quad \left(\hat{=}\, e^{iA} E e^{-iA} = E+1\right)$
  - magnetic flux through plaquette s: $\sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}^{(**)}$ $\left(\hat{=}\, e^{iA_{ij}} \cdots e^{iA_{li}}\right)$

  electric and magnetic flux are gauge invariant

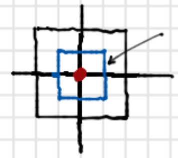
- gauge invariant Hamiltonian

$$\hat{H} = -\gamma \sum_{\text{links}} \overset{(*)}{\sigma_{x,ij}} - \lambda \sum_{\square} \overset{(**)}{\sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}}$$

- system supports quantum phase transition driven by $\tau \equiv \lambda/\gamma$
  1) confining phase $\tau \lesssim 1$
  2) topological phase $\tau \gtrsim 1$

1) Confining phase

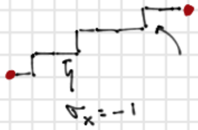
- Define 'charge density op': $\hat{\rho}_i = \prod_j \sigma_{x,ij}$
  Charge is locally conserved by $\hat{H}$: $[\hat{H}, \hat{\rho}_i] = 0$



- Ground state in charge neutral sector: $\forall i \;\; \hat{\rho}_i |\Psi\rangle = 0$:
  $\sigma_{x,ij} = 1$ globally

- Q: What is ground state in sector of Hilbert space with two charges sitting at i, j?

A: $(\lambda = 0)$

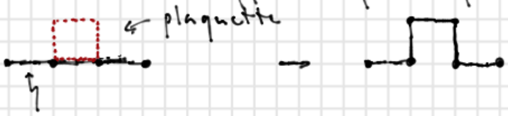


a minimal electrix flux line. Costs energy $2\gamma\, d(i,j) = \sigma\, d(i,j)$
Manhattan
string tension

Energy grows linearly with distance: confinement

- Q: What is the effect of the flux term on this

A:



$\sim$ string fluctuations $\sigma \to 2\gamma - \lambda^2/4\gamma$
strength of pert.
excitation energy

magnetic flux term softens string tension

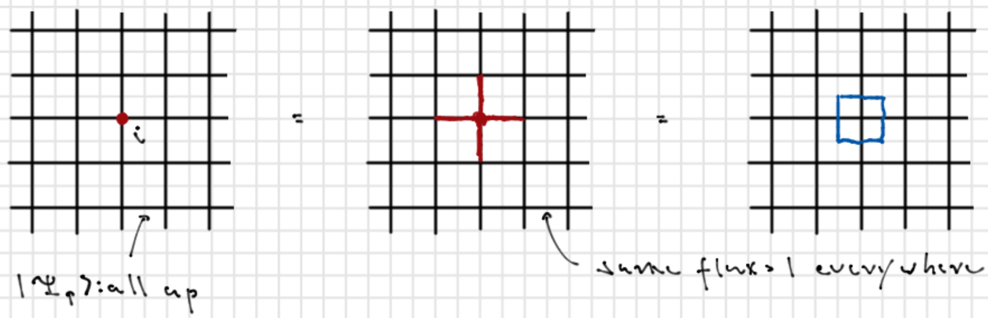
2) Spin liquid phase

- Consider limit $\gamma = 0$. Ground state has flux $\prod_{\square} \sigma_z \dots \sigma_z$ on each plaquette. This is an implicit characterization.

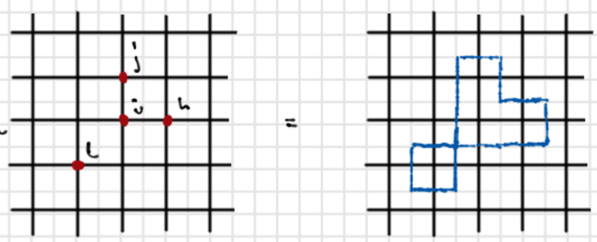
- Q: How does the ground state in a sector of fixed charge look like?

A: Start from all spins $\sigma_{z,ij} = 1$ state $|\Psi_\uparrow\rangle$. This is not a charge eigenstate

$\hat{p}_i$



$\hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l$



ground state: state of all equal weight superposition of closed strings (cf. BCS), a string net condensate

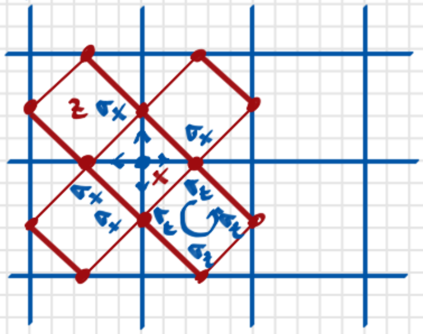
Corrolary: charges totally deconfined.

Q: How does this system relate to surface code?

i) Implement charge constraint (e.g. $\rho_i = 0$) by contribution to Hamiltonian

$$H_c = c \sum_i \prod_{+} \sigma_{x,i} .$$

ii) Map lattice gauge (link) Hamiltonian to equivalent Hamiltonians defined on sites of rotated lattice:



$$H = c \sum_{P_x} \prod_{\circlearrowleft} \sigma_x + \lambda \sum_{P_z} \prod_{\circlearrowleft} \sigma_z$$

Q: Is the ground state unique?

A: Def.: V: no. of vertices, E: no. of edges, F: no. of faces

Compactify surface (for simplicity), e.g. . Counting:

E d.o.f. (the spins)

$-(V-1)$ charge constraints (-1 is overall charge neutrality)

$-(F-1)$ flux constraint (-1 is overall flux neutrality)

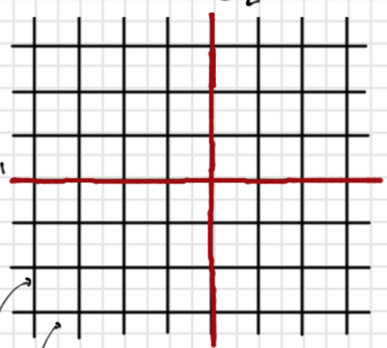
$\underbrace{-F+E-V}_{(-)\text{Euler-Characteristic}} - 2$ qubit d.o.f. remain

(−) Euler-Characteristic $\chi = 2-2g$. Surface genus g.

$\leadsto$ ground state degeneracy: 2^{2g}. A hallmark of topological matter.

Q: How do we characterize different ground states?

A:

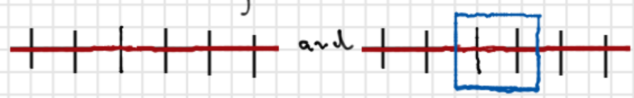


assume periodic boundary conditions a torus, T^2

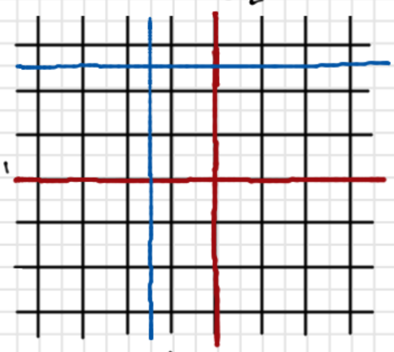
$\sigma_z^a = \prod_{C_a} \sigma_{z,ij}$.

or any other curve winding around T^2.

Claim: $\{\sigma_z^1, \sigma_z^2\}$ are topological invariants of each charge sector. E.g.



same C^1



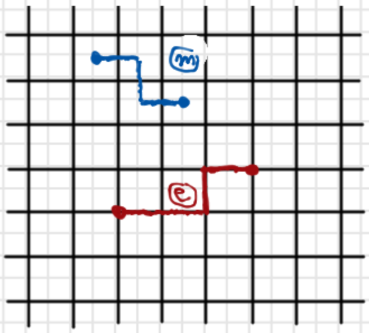
Invariants changed by nonlocal operators.

$\sigma_x^a = \prod_{C'_a} \sigma_{x,ij}$ $\qquad [\sigma_x^a, \sigma_z^a]_+ = 0$

Global qubits $\{\sigma_x^a, \sigma_z^a\}$ provide topological characterization of ground state $\leadsto$ topological quantum computation.

Q: What are excitations of the system?

A:



m: a 'magnetic' excitation $= \prod_{\substack{\text{all links} \\ \text{cut by string}}} \sigma_{x,ij}$

changes flux of terminal plaquettes. Costs energy 2λ.

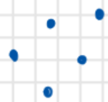
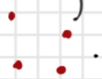
e: an 'electric excitation $= \prod_{\substack{\text{all links along} \\ \text{path}}} \sigma_{z,ij}$

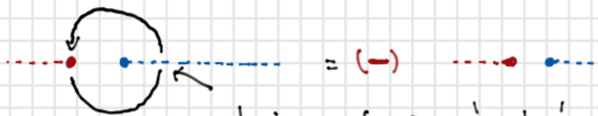
changes charge at terminal points. If system contains 'chemical potential' $\mu \cdot \sum_i \hat{p}_i$, energy cost 2μ.

Think of e, m as quantum particles forming on top of (closed) string ground state

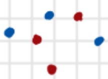
Q: What is the statistics of these particles?

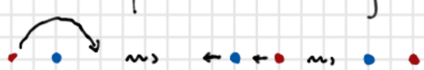
A: Find out what happens as we braid them around one other.

$\rightsquigarrow$ nothing happens  : a system of bosons. The same with .

However: 

string of σ_x 'cuts' through one of σ_z: a minus sign

 are semions relative to each other. Note: particle exchange: a π-rotation



	π-exch.	2π-braid
fermions	-1	1
semions	i	-1

Something interesting happens if we fuse m and e into a composite excitation:

 $\rightsquigarrow$  are fermions relative to each other. Have generated fermions as

- emergent particles of gauge theory
- terminal excitations of strings (cf. Jordan-Wigner in 1d)
- particles obeying strict parity conservation

$\Bigg\} \rightsquigarrow$ conceptual proximity to string theory